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ON STOCHASTIC ANALYSIS APPROACHES FOR COMPARING COMPLEX SYSTEMS

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Abstract— We compare two probabilistic approaches that have been proposed for the study and comparison of data obtained from experiment or simulations of a dynamical systems model. Both the approaches aim to carry out spectral analysis using a time-series data that is generated by a dynamical system. The first approach is based on constructing the so-called harmonic averages from the time-series data [1] and the second is an approach based on construction of a Markov chain from the data [2]. Results of the comparison using time-series data from a combustion model is provided.

I. INTRODUCTION

Results from experiments are typically available in the form of time-series measurements. Physical models of the experiment, that may include noise as an input, can also be simulated to obtain time-series. Because of presence of stochastic driving noise or because of sensitivity due to the presence of chaotic dynamics, the question of "When is a model good enough?" is as difficult as it is important.

One crude statistical approach to comparison is to compare the stationary distribution (histogram) representing the time-averaged properties of the data. However, such an approach ignores the spectrum of the signal. For instance, two sine waves with same amplitude but different frequencies have the same histogram but different spectral characteristics. In the recent pioneering work of [1], a formalism for comparing the asymptotic dynamics resulting from different dynamical models is provided. The formalism is based upon spectral properties of Koopman operator. The idea is to construct harmonic averages (in addition to time averages) and use these to analyze the spectral characteristics. This generalizes the concept of comparing the stationary distribution alone.

The spectral analysis of time-series data generated from a dynamical system, now for the purpose of recovering cyclic behavior, is pursued in the work of [2]. In these papers, the formalism is based upon Perron-Frobenius operator, that (in suitable functional analytic settings) is the adjoint to the Koopman operator [3]. A set-oriented numerical method is described to numerically approximate the infinite-dimensional Frobenius-Perron operator with a finite-dimensional Markov chain matrix. Modulus 1 eigenvalues of these matrix are used to extract the cyclic behavior.

The objective of this paper is to compare and contrast the two approaches. In particular, we are interested in

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determining the relationship between the spectral analysis afforded by the two approaches. In studying the connections between the two approaches, we distinguish between the infinite-dimensional (continuous) formalism and the finite-dimensional implementation (with a choice of finite partition). For invertible mappings, the relationship for the former is studied in [1]. In particular, the harmonic averages are shown to yield an eigenmeasure for the P-F operator.

However, the relationship between the two approaches is not clear once one takes a finite partition for the purpose of computational implementation. In this paper, we investigate the relationship between the harmonic averages (based on Koopman formalism [1]) and the eigenmeasures (based on P-F formalism [2]) for the finite-dimensional problem. From our intuition of the continuous problem, it is perhaps reasonable to expect the two to be related (for fine enough partition). It is the purpose of the paper to demonstrate this. The results are obtained using calculations that explicitly account for partition. We show that choice of indicator functions as basis functions is a natural counterpart of constructing P-F matrix with co-ordinates that correspond to measures defined on (σ-algebra) of the partition. Results are illustrated with the help of examples, including an example from a UTRC combustion model.

The outline of this paper is as follows. In Section II, we present the notation and definitions for the stochastic objects under study. In Section III, we present the formalism pertaining to the spectral analysis of the continuous infinite-dimensional problem. A discussion of both the Koopman and the P-F formalism and their mutual relationship is provided. In Section IV, we present the results now for a finite partition. The approach is novel and constructive, with explicit formulas, so as to be useful to the practitioners. In Section V, we present two examples, one academic and other using a nonlinear model of the UTRC combustion rig, to demonstrate the finite-dimensional approach of this paper. Finally in Section VI, we draw up some conclusions, including our assessment of the merits of the two approaches and directions for future work.

II. PRELIMINARIES AND NOTATION

We assume $X \subset \mathbb{R}^n$ and consider discrete dynamical systems of the form

$$x_{n+1} = T(x_n), \tag{1}$$

where $T : X \subset \mathbb{R}^n \to X$ is an invertible smooth mapping (diffeomorphism) that depends additionally on parameters. To aid probabilistic treatment of the problem, we introduce

some notation from the field of Ergodic theory [4], [3]. We denote by $\mathscr{B}(X)$ the Borel σ-algebra on X and define basic stochastic objects. We use $\mathscr{M}$ to denote the vector-space of bounded complex valued measures on $\mathscr{B}$.

Definition 1 (Perron-Frobenius (P-F) operator):
$\mathbb{P} : \mathscr{M} \to \mathscr{M}$ is defined by

$$P[\mu](A) = \int_X \delta_{T(x)}(A) d\mu(x) = \mu(T^{-1}(A)), \quad (2)$$

where T is the map in Eq. (1).
Here, $\delta_{T(x)}(A)$ is the **stochastic transition function**, which measures the probability that the point x reaches the set A under one iterate of the map T.

Definition 2 (Invariant measures): are the fixed points of the P-F operator $\mathbb{P}$ that are additionally probability measures.

It is known that the P-F operator has atleast one eigenvalue at 1 [3], [5], i.e., any map T admits atleast one invariant measure (denoted as μ_1).

Definition 3 (Koopman operator): $\mathbb{U} : \mathscr{F} \to \mathscr{F}$ is defined by

$$Uf(x) = f(Tx), \quad (3)$$

where $x \in X$.
For the choice of $\mathscr{F} = L^2(X, \mathscr{B}, \mu_1)$ as a Hilbert space, it is easy to see that

$$\|Uf\|^2 = \int_X |f(Tx)|^2 d\mu_1(x) = \|f\|^2, \quad (4)$$

where we have used the fact that μ_1 is an invariant measure.

If instead of the measure space, the P-F operator is defined on the function space of densities ($L^1(X)$), it can be shown that the P-F and the Koopman operator (defined on $L^\infty(X)$) are adjoint [3]. In this paper, we assume that the Ergodic hypothesis applies.

III. CONTINUOUS PROBLEM

A. Koopman formalism

By Eq. (4), the Koopman operator U is a unitary operator. Hence the spectrum $\sigma(U)$ and in particular, the eigenvalues of U lies on the unit circle. Denote by $E_\lambda = \{f \in \mathscr{H} : Uf = \lambda f\}$, where $|\lambda| = 1$ and $\mathscr{H}$ denotes the Hilbert space obtained after complexification of the space $L^2(X, \mathscr{B}, \mu_1)$. For $\lambda \in \mathbb{C}$ with absolute value 1, define the harmonic average by

$$P_T^\lambda g(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} (\lambda U)^k g(x), \quad (5)$$

where $(\lambda U)^k g(x) = \lambda^k g(T^k x)$. The operator P_T^1 is the special case of the above and corresponds to the time average. Since λU is unitary for $|\lambda| = 1$, the following theorem is a simple corollary of the mean Ergodic theorem in Hilbert space (Theorem 1.2 in [5]).

Theorem 4: $P_T^\lambda : \mathscr{H} \to E_{\bar{\lambda}}$ is the projection operator onto eigenspace $E_{\bar{\lambda}}$ corresponding to eigenfunction of U with eigenvalue $\bar{\lambda}$.
We remark that if $|\lambda| \leq 1$ and $\lambda \notin \sigma(U)$ then the harmonic average $P_T^\lambda g(x)$ converges to zero. In fact, the harmonic average is non-zero only on atmost a countable set of λ

that correspond to the eigenvalues of the operator U [6], [1]. Thus, constructing harmonic averages with such λ can be used as a method for distinguishing dynamical systems as is advocated in [1]. For $\lambda = 1$, the operator P_T^1 leads to non-zero time-averages (for suitable choice of basis function g) because $1 \in \sigma(U)$ and is in fact an eigenvalue.

B. Perron-Frobenius formalism

If the mapping $T : X \to X$ (with P-F opertor $\mathbb{P}$) is assumed invertible, the P-F operator for the inverse mapping $T^{-1} : X \to X$ is given by $\mathbb{P}^*$ where, $\mathbb{P}\mathbb{P}^* = \mathbb{P}^*\mathbb{P} = 1$, i.e., $\mathbb{P}$ is unitary.

By the unitary property, the spectrum of such an operator lies on the unit circle. The individual modulus 1 eigenvalues and their eigenfunctions can be used to extract information on cyclic behavior [2] present in the solutions of the dynamical system (with map T). In particular, the eigenmeasure corresponding to eigenvalue 1 (that always exists) represents the invariant measure. Likewise an eigenvalue -1, if it exists, would point to the presence of cyclic behavior with period 2. The behavior may be as simple as a period 2 orbit $\{x_0, T(x_0), x_0, T(x_0), ...\}$ or it may be complex where the trajectory is chaotic but moves with a period 2 between two disjoint sets, i.e.,

$$T^k(x) \in \begin{cases} A_0 & : \quad k \bmod 2 = 0 \\ A_1 & : \quad k \bmod 2 = 1 \end{cases} \quad (6)$$

The eigenfunction (for eigenvalue -1) can be used to distinguish between the two cases.

We remark that T need not be an invertible map in Eq. (2) for the definition of the P-F operator. Indeed, for a non-invertible T, $T^{-1}(A)$ is simply the pre-image set consisting of all the points that lie in the set A after one iterate of mapping T. In this case, $\mathbb{P}$ is generally not unitary and the spectrum is only known to lie inside the unit circle.

Non-invertibility of T is not the only way for $\mathbb{P}$ to be non-unitary. Even discretization with finite partitions may lead to a non-unitary Markov chain $\mathbb{P}$ as will be discussed in Section IV.

C. Harmonic averages and P-F eigenmeasures

The two formalisms suggest two approaches for comparing dynamical systems: (1) using harmonic averages, and (2) using eigenmeasures of the P-F operator. In fact, the two approaches are equivalent and yield the same information. For the particular case of eigenvalue 1, this follows from the equivalence of the space and the time-averages (under suitable Ergodic hypothesis), i.e.,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} U^k g(x) = \int_X g d\mu_1 \quad (7)$$

independent of x (in a.e. sense). More generally, the following theorem is shown in [1]:
Theorem 5: For a.e. x,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} (\lambda U)^k g(x) = \int_X g d\mu_{\bar{\lambda}, x}, \quad (8)$$

where $\mathbb{P}\mu_{\lambda,x} = \bar{\lambda}\mu_{\bar{\lambda},x}$. Moreover, $\mu_{\bar{\lambda},Tx} = \bar{\lambda}\mu_{\bar{\lambda},x}$.
We have the following corollary:

Corollary 6: Suppose $\bar{\lambda} \in \sigma(\mathbb{P})$ then

1) $|\mu_{\bar{\lambda},x}| = \mu_1$
2) if $x \in A$ then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} (\lambda U)^k \kappa_A(x) = \mu_1(A). \qquad (9)$$

D. Comparison of dynamics

One possible approach to comparing two dynamical systems could be to compare the time-averages or equivalently the invariant measures. Such a comparison, however, fails to account for the spectral characteristics of the possible behavior. From the results of this section, two equivalent directions to obtain spectral information would be 1) to construct harmonic averages or 2) to compute eigenmeasures of the P-F operator. Either of these yield objects that can be used to compare (solutions of) dynamical systems. A appropriately defined distance function between two sets of harmonic averages (eigenmeasures) yield the so-called psuedometric for comparison.

IV. DISCRETE PROBLEM FOR A FINITE PARTITION

In practice, one begins with a time-series data generated from an experiment or from simulating a model. While the phase space co-ordinates for a model is known, we assume that the time-series from an experiment has been embedded in to a suitable phase space $X \subset \mathbb{R}^m$. This may be accomplished by using a simple delay co-ordinate technique together with an embedding theorem [7]. We assume a finite time-series data $\{x_0, T(x_0), T^2(x_0), ..., T^{N-1}(x_0)\}$ of length N that is embedded in a suitable space $X \subset \mathbb{R}^m$. We assume that N is large enough to take good averages. We carry out spectral analysis of this time-series data using the discrete version of the two approaches. To obtain a discrete problem, we work with finite partitions. Instead of the phase space X, we consider $X' = \{D_1, \cdots, D_L\}$, a finite partition of X (such that $\cup_j D_j = X$).

A. Harmonic averages for indicator functions

For $X' = \{D_1, \cdots, D_L\}$, let κ_j denote the indicator function with support on D_j. For a given normalized frequency $\omega \in [-0.5, 0.5]$, denote $\lambda = e^{i2\pi\omega}$. Take the harmonic average for each κ_j to obtain a complex value for D_j as

$$\kappa_{j,\lambda(\omega)}^* = \frac{1}{N} \sum_{k=0}^{N-1} (\lambda)^k \kappa_j(T^k(x_0)). \qquad (10)$$

In [1] the harmonic averages, thus constructed, are used to define a pseudometric. In particular, the psuedometric measures the distance between the harmonic average constructed with the data from the model against the data from the experiment. In this approach, a frequency ω needs to be picked first. The authors suggest using DFT of the signal to obtain the frequency. The advantage of this approach is that the harmonic averages are straightforward to compute.

B. P-F matrix for finite partitions

Recall that $\mathscr{B}(X)$ denotes the (Borel) σ-algebra defined *on* X, $\mathscr{M}(X)$ denotes the measure space *on* X and $\mathbb{P}: \mathscr{M}(X) \to \mathscr{M}(X)$. For the discrete problem, one instead considers $X' \doteq \{D_1, \cdots, D_L\}$, a finite partition of X (such that $\cup_j D_j = X$). Now, instead of a Borel σ-algebra, consider a σ-algebra is all possible subsets of X'. A real-valued measure μ_j is defined by ascribing to each element D_j a real number. This allows one to identify the associated measure space with a finite-dimensional real vector space $\mathbb{R}^L$. A given mapping $T: X \to X$ defines a stochastic transition function $\delta_{T(x)}(\cdot)$. This function can be used to obtain a coarser representation (denoted by $\mathbb{P}': \mathbb{R}^L \to \mathbb{R}^L$) for the continuous P-F operator $\mathbb{P}$ as follows: For $\mu' = (\mu_1', \cdots, \mu_L') \in \mathbb{R}^L$, we define a measure on X as

$$d\mu(x) = \sum_{j=1}^{L} \mu_j' \kappa_j(x) \frac{dm(x)}{m(D_j)}, \qquad (11)$$

where m is the Lebesgue measure and κ_j denotes the indicator function with support on D_j. The P-F matrix can now be obtained by

$$\begin{aligned} v_i' &= \mathbb{P}'[\mu'](D_i) \\ &= \sum_{j=1}^{L} \int_{D_j} \delta_{T(x)}(D_i) \mu_j' \frac{dm(x)}{m(D_j)} = \sum_{j=1}^{n} \mu_j' \mathbb{P}_{ij}', \quad (12) \end{aligned}$$

where

$$\mathbb{P}_{ij}' = \frac{m(T^{-1}(D_i) \cap D_j)}{m(D_j)}, \qquad (13)$$

m being the Lebesgue measure. We note that the resulting matrix is a Markov chain. One can use a Monte-Carlo approach to compute the individual entries $\mathbb{P}_{ij}'$. In the simplest settings, one begins with M "initial conditions" uniformly distributed in the box D_j and use the mapping T to flow these initial conditions; each initial condition is used to represent a volume in the phase space. The entry $\mathbb{P}_{ij}$ is then approximated by the fraction of initial conditions that are in the box D_i after one iteration of the mapping.

We may also extend this Monte-Carlo approach to the case where a sufficiently long time-series data $\{x_0, T(x_0), T^2(x_0), ..., T^{N-1}(x_0)\}$ of length N is given. The number of "initial conditions" in box j are then given by

$$\sum_{l=0}^{N-1} \kappa_j(T^l(x_0)) \qquad (14)$$

and the ij-entry of the P-F matrix is the fraction of these initial conditions that are in set D_i after one iterate of map T. This leads to an explicit formula

$$\mathbb{P}_{ij}' = \frac{1}{\sum_{l=0}^{N-1} \kappa_j(T^l(x_0))} \sum_{k=0}^{N-1} \kappa_j(T^k(x_0)) \kappa_i(T^{k+1}(x_0)) \quad (15)$$

given entirely in terms of indicator functions. Eigenvalues of this matrix provides for a spectral analysis of the time-series data.

One may construct the psuedometric for comparison by using the individual eigenmeasures. For a given partition, one looks for eigenvalues "close" to the unit circle. These

eigenvalues yield information on the behavior that persists at the coarse scale of the partition. The spectral comparison between the data from the model and the experiment involves comparing the eigenvalues as well as comparing the eigenmeasures. In particular, if there is a single eigenvalue (apart from the unity eigenvalue) close to the unit circle, one deems the model to validate the experiment if the eigenvalue match and so do the eigenspace. The advantage of this approach above the former one is that one need not guess the frequency ω.

C. Harmonic averages and P-F eigenmeasures

We begin by showing the relationship between time-averages and the invariant measure (with $\lambda = 1$). Set

$$\mu^1[j] = \kappa^*_{j,1} = \frac{1}{N}\sum_{k=0}^{N-1}\kappa_j(T^k(x_0)). \qquad (16)$$

Using Eq. (15), we have

$$\mathbb{P}'_{ij}\mu^1[j] = \frac{1}{N}\sum_{k=0}^{N-1}\kappa_j(T^k(x_0))\kappa_i(T^{k+1}(x_0)), \qquad (17)$$

and taking a sum over j, we obtain

$$\begin{aligned}
\sum_{j=1}^{L}\mathbb{P}'_{ij}\mu^1[j] &= \frac{1}{N}\sum_{k=0}^{N-1}\kappa_i(T^{k+1}(x_0))\sum_{j=1}^{L}\kappa_j(T^k(x_0))\\
&= \frac{1}{N}\sum_{k=0}^{N-1}\kappa_i(T^{k+1}(x_0))\cdot 1\\
&= \mu^1[i], \qquad (18)
\end{aligned}$$

i.e., the vector $\mu^1 \in \mathbb{R}^L$ is in fact the invariant measure. For the more general case, set

$$\mu^{\bar{\lambda}}[j] = \kappa^*_{j,\lambda} = \frac{1}{N}\sum_{k=0}^{N-1}\lambda^k\kappa_j(T^k(x_0)). \qquad (19)$$

We have the following theorem:

Theorem 7: Suppose λ is an eigenvalue of $\mathbb{P}'$ with unit modulus then for a suitably fine partition,

$$\mathbb{P}'\mu^{\bar{\lambda}} = \bar{\lambda}\mu^{\bar{\lambda}}, \qquad (20)$$

where $\mu^{\bar{\lambda}}$ is given by Eq. (19).

Proof:

$$\mathbb{P}'_{ij}\mu^{\bar{\lambda}}[j] = \frac{1}{N}\sum_{k=0}^{N-1}\lambda^k\kappa_i(T^{k+1}x_0)\frac{\sum_{l=0}^{N-1}\lambda^{l-k}\kappa_j(T^lx_0)\kappa_j(T^kx_0)}{\sum_{p=0}^{N-1}\kappa_j(T^px_0)} \qquad (21)$$

Taking a summation over index j,

$$\sum_{j=1}^{L}\mathbb{P}'_{ij}\mu^{\bar{\lambda}}[j] = \frac{1}{N}\sum_{k=0}^{N-1}\lambda^k\kappa_i(T^{k+1}x_0)\sum_{l=0}^{N-1}\lambda^{l-k}\sum_{j=1}^{L}\frac{\kappa_j(T^lx_0)\kappa_j(T^kx_0)}{\sum_{p=0}^{N-1}\kappa_j(T^px_0)} \qquad (22)$$

For fixed k, suppose $T^k x_0 \in D_{J(k)}$ then

$$\sum_{j=1}^{L}\mathbb{P}'_{ij}\mu^{\bar{\lambda}}[j] = \frac{1}{N}\sum_{k=0}^{N-1}\lambda^k\kappa_i(T^{k+1}x_0)\frac{\sum_{l=0}^{N-1}\lambda^{l-k}\kappa_{J(k)}(T^lx_0)}{\sum_{p=0}^{N-1}\kappa_{J(k)}(T^px_0)} \qquad (23)$$

Using the part (2) of Corollary 6 above,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{l=0}^{N-1}\lambda^{l-k}\kappa_{J(k)}(T^lx_0) = \mu_1(D_{J(k)}), \qquad (24)$$

the invariant measure for the continuous problem. Next, by Eq. (16)

$$\frac{1}{N}\sum_{p=0}^{N-1}\kappa_{J(k)}(T^px_0) = \mu^1[J(k)]. \qquad (25)$$

Now, for N long enough and partition fine enough, it is true that $\mu_1(D_{J(k)}) \approx \mu^1[J(k)]$ [2] and we have

$$\frac{\sum_{l=0}^{N-1}\lambda^{l-k}\kappa_{J(k)}(T^lx_0)}{\sum_{p=0}^{N-1}\kappa_{J(k)}(T^px_0)} \to 1. \qquad (26)$$

Substituting Eq. (26) into Eq. (23),

$$\sum_{j=1}^{L}\mathbb{P}'_{ij}\mu^{\bar{\lambda}}[j] \to \frac{\bar{\lambda}}{N}\sum_{k=0}^{N-1}\lambda^{k+1}\kappa_i(T^{k+1}x_0) = \bar{\lambda}\mu^{\bar{\lambda}}[i] \qquad (27)$$

and this proves the theorem. ∎

V. EXAMPLES

A. Period-4 oribit with noise
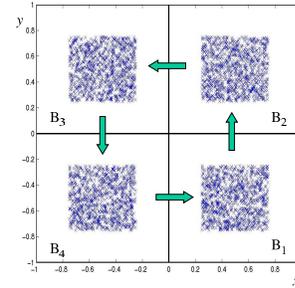
Fig. 1.   Phase space plot of the period-4 time-series data.

Fig. 1 depicts a time-series consisting of a period-4 orbit

$$\{x_1, x_2, x_3, x_4\} \qquad (28)$$

perturbed with bounded random noise; $x_1 = [0.5, -0.5]$, $x_2 = [0.5, 0.5]$, $x_3 = [-0.5, 0.5]$, $x_4 = [-0.5, -0.5]$. The time-series data with $N = 4000$ points is embedded in $B \doteq [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ and $B' = \{B_1, B_2, B_3, B_4\}$ denote a finite partition of B consisting of four boxes (see Fig. 1). For this partition and the time-series, the underlying dynamical system T satisfies the *deterministic* and cyclic relationship

$$\begin{aligned}
T(B_i) &\subset B_{i+1} \quad \text{for } i = 1, 2, 3\\
T(B_4) &\subset B_1. \qquad (29)
\end{aligned}$$

The corresponding P-F matrix corresponds to a deterministic [3] and unitary transfer matrix

$$\mathbb{P}' = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (30)$$

with eigenvalues $\{1, i, -1, -i\}$. We consider the harmonic averages for individual eigenvalues. Since noise is small relative to the size of the box, a very good approximation for harmonic average (for large N) is

$$\kappa_{j,\lambda}^* = \frac{1}{4} \sum_{k=0}^{3} \lambda^k \kappa_j(T^k(x_0)). \qquad (31)$$

There exist four possibilities for the choice of x_0: $x_0 \in B_j$ for $j = 1, 2, 3, 4$.

It is easily seen that for $\lambda = 1$, $\kappa_{j,1}^* = \frac{1}{4}$ independent of the choice of x_0. Moreover, this is precisely the invariant measure - eigenmeasure for the matrix in Eq. (30) with eigenvalue $\lambda = 1$. The eigenmeasures for other eigenvalues of the P-F matrix in Eq. (30) may also be consistently obtained and used for constructing a pseudometric for spectral comparison. For instance, the eigenmeasure corresponding to the eigenvalue $\lambda = \pm i$ picks up the period-4 cyclic behavior and the P-F matrix and hence its eigenmeasures are independent of the exact realization of the noise.

However, for eigenvalues $\lambda \neq 1$, the harmonic average $\kappa_{j,\lambda}^*$ depends upon the choice of initial condition. Its value for $\lambda = i$ is summarized in the following table

$$
\begin{array}{c|cccc}
x_0 \in & B_1 & B_2 & B_3 & B_4 \\
\hline
\kappa_1^* & 1 & -i & -1 & i \\
\kappa_2^* & i & 1 & -i & -1 \\
\kappa_3^* & -1 & i & 1 & -i \\
\kappa_4^* & -i & -1 & i & 1
\end{array} . \qquad (32)
$$

It is however easy to see that these functions differ from each other only by a complex factor (suitable power of λ). Moreover, each of these functions lie in the complex eigenspace corresponding to the eigenvalue $\bar{\lambda} = -i$. As a result, the spectral analysis of the P-F matrix constructed using a partition provides for pseudometrics that are identical to ones obtained using harmonic averages corresponding to indicator functions on the partition. For the realization of the noise used in constructing the time-series of Fig 1, there was only $O(10^{-16})$ difference in the pseudometrics constructed using the two approaches.
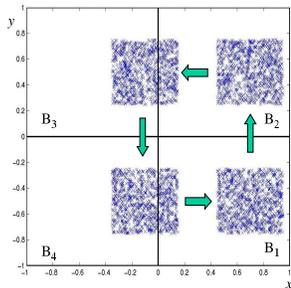
Fig. 2.   Phase space plot of the perturbed period-4 time-series data.

We remark that the construction of pseudometric depend upon the selection of the partition. As an example, consider the period-4 time-series shown in Fig 2 but now with a partition that is not fine enough to resolve the cyclic behavior. In this case, the P-F matrix corresponding to the partition is

not deterministic; depending upon the instantaneous value of noise there may be a transition from $B_2 \to B_2$ or $B_2 \to B_3$. Numerically, the P-F matrix was found to have eigenvalues $\{1, -0.006 + 0.727i, -0.006 - 0.727i, -0.516\}$, i.e., it is not unitary. Figure 3 compares the absolute value of the normalized error between the eigenfunctions obtained with the two approaches. In order to construct the harmonic averages, the eigenvalues $\{1, i, -i, -1\}$ were used. The harmonic averages are compared against the eigenfunction with the closest corresponding eigenvalue of the P-F matrix.
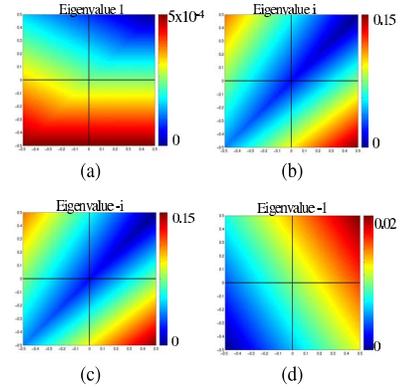
Fig. 3.   Normalized error between the (suitably normalized) eigenfunctions obtained using the two approaches for the time-series and the partition of Fig. 2.

B. Time-series data from combustion model

In this section, we compare the results of the spectral analysis carried out using two approaches for the time-series data that is generated from the combustion model first presented in [1]. The model is a two-state oscillator with a saturation nonlinearity in feedback and is driven by a broadband stochastic disturbance. The equations are

$$
\begin{aligned}
x(k) &= (-\alpha + \cos(\omega_0 T_s))x(k-1) - \sin(\omega_0 T_s)y(k-1), \\
y(k) &= \sin(\omega_0 T_s)x(k-1) + (-\alpha + \cos(\omega_0 T_s))y(k-1) \\
&\quad + K_3 h(K_2 x(k-D)) + K_1 n(k-1), \qquad (33)
\end{aligned}
$$

where the model parameters are $T_s = 0.0005, \alpha = 0.03, \omega_0 = 2\pi \cdot 207, K_1 = 1, K_2 = 2000, K_3 = 0.0525, D = 10$, h is a saturation (at 1) nonlinearity, and $n(\cdot)$ is taken to be a random noise uniformly distributed in the interval $[-0.1, 0.1]$. Fig. 4 plots a $N = 10000$ length time-series obtained with this model and initial conditions $x(0) = 1, y(0) = 0$. Also shown
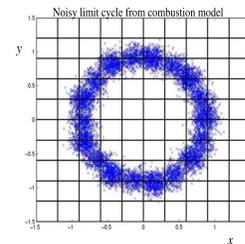
Fig. 4.   Phase space plot of the time-series data generated with model Eq. (33).

is the 10×10 partition of the phase space $[-1.5, 1.5] \times [-1.5, 1.5]$ that is used for carrying out the spectral analysis: for defining the indicator functions for harmonic averages and for defining the coordinates of the 100×100 P-F matrix. Fig. 5 shows the eigenvalues of this matrix. The figure shows
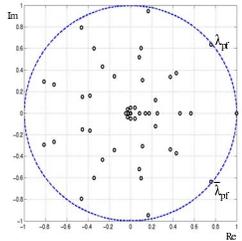
Fig. 5. Eigenvalues of the P-F matrix corresponding to the time-series data and the partition of Fig. 4.

that for the coarse partition considered, the eigenvalues are only bounded by unit circle. Apart from the unity eigenvalue, there is an eigenvalue denoted λ_{pf} close to the unit circle that captures the frequency of the limit cycle. In fact, there is a band of eigenvalues close to the unit circle all related to the superharmonics of λ_{pf}.

Fig. 6 compares the time-average and the eigenfunction corresponding to the eigenvalue 1 (invariant measure). On 10×10 partition, the maximim difference between the two is of $O(10^{-3})$. The invariant measure also shows the attractor to be a noisy limit cycle [1].
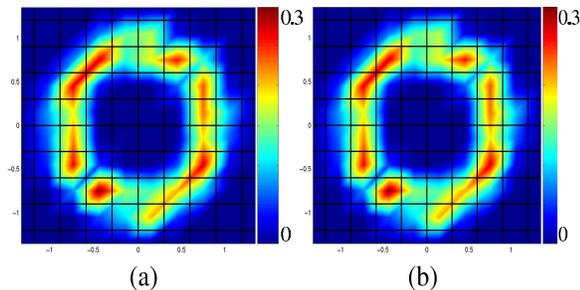
Fig. 6. Invariant measure constructed using (a) harmonic averages with indicator functions and (b) normalized eigenmeasure for eigenvalue 1 of the 100×100 P-F matrix.

Next, a discrete Fourier transform of the signal shows a dominant peak at frequency of $2\pi \cdot 222.2$ (corresponding $\lambda = 0.766 + i0.643 \doteq \lambda_h$). According to the suggestion in [1], this is used to construct harmonic averages for 100 indicator functions that are defined by the 10×10 partition. This is compared against the suitably normalized eigenmeasure corresponding to the eigenvalue $\lambda = 0.758 + i0.637 \doteq \lambda_{pf}$ of the P-F matrix. Fig. 7 depicts the real and the imaginary parts of this eigenfunction. The eigenfunction matches the harmonic averages very closely. The normalized error between the two approaches is ≤ 0.05.

We finally note that we have not compared the absolute value of the eigenfunctions. For modulus 1 eigenvalues, the absolute value of the eigenmeasure is the invariant measure and hence does not provide additional information.
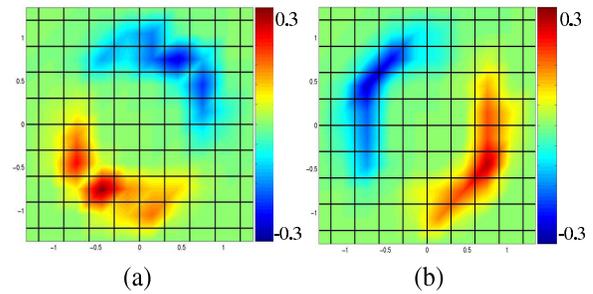
Fig. 7. (a) Real and (b) Imaginary parts of the eigenmeasure corresponding to the eigenvalue λ_{pf} of the 100×100 P-F matrix.

VI. CONCLUSIONS

In this paper, we have provided a comparison between two approaches for comparing dynamical systems. With the aid of theory and examples, we show the two approaches to be equivalent. The P-F matrix approach provides for a complete spectral analysis of the data (for a given partition). If the eigenvalues of this matrix are known then harmonic average is a straightforward method to obtain the eigenfunctions of the P-F matrix. The advantage is that this avoid an eigenvalue calculation of a matrix. The disadvantage is that it requires apriori knowledge of the eigenvalue. Moreover, for long time-series, the harmonic averages are sensitive with respect to λ: it is generically zero. The harmonic average also depends upon the choice of initial condition. However, this is a minor issue as the harmonic average for two initial conditions differ only by a complex factor.

In future work, we would consider the problem of integrating the comparison problem with model identification problem from experimental data. The spectral analysis results presented here are seen as the first step in that direction.

VII. ACKNOWLEDGEMENT

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