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Converse theorem for almost everywhere stability using Lyapunov measure

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***Abstract*—**
In our recent paper [1][2], *Lyapunov measure* **is introduced as a new tool for verifying almost everywhere stability of an invariant set in dynamical systems and continuous mapping. In this paper we show that the existence of Lyapunov measure is both necessary and sufficient for almost everywhere stability. The necessary and sufficient condition for almost everywhere stability using Lyapunov measure is analogous to necessary and sufficient condition for asymptotic stability in linear system. In particular the finite dimensional matrix Lyapunov equation for verifying stability in linear systems is replaced by infinite dimensional linear equation for verifying almost everywhere stability of an invariant set in nonlinear systems.**

I. INTRODUCTION

The problem of stability is one of the central topic of interest in control theory. The ultimate goal is to find verifiable conditions to guarantee stability of a given system. For finite dimensional linear systems such conditions exist, one of them is in the form of Lyapunov equation [3]. Given the complexity of behavior possible in nonlinear system [4], the task of obtaining such verifiable condition for nonlinear systems is not easy. The problem is more complex given various possible notions of stability for nonlinear systems. In the recent work by Rantzer [5], weaker notion of almost everywhere stability and criteria for verifying this notion is introduced in the form of density function. It is shown that existence of density function is sufficient for the global attractive property of an equilibrium solution from almost every with respect to Lebesgue measure initial conditions in the phase space. Density function is shown to be dual to Lyapunov function. Converse theorem for density function also exists [6][7]. Under the assumption that the equilibrium point is locally stable (in Lyapunov sense), existence of density function is also shown to be necessary for almost everywhere stability of an equilibrium point.

In our paper [1][2], Lyapunov measure is introduced to study almost everywhere stability of an invariant set in dynamical system or continuous mapping. Existence of Lyapunov measure is show to be sufficient for a.e. stability of an attractor set. Lyapunov measure is also shown to be dual to Lyapunov function. In this paper we introduce new definition of stability of an invariant set which is referred to as *almost everywhere uniformly stable*. The main result of this paper shows that the existence of Lyapunov measure is both necessary and sufficient for almost everywhere uniform stability of an invariant set. More importantly the necessary and sufficient condition on Lyapunov measure that we have are analogous to necessary and sufficient condition for stability in linear system. In particular the finite dimensional Lyapunov matrix equation for verifying stability in linear system is replaced by infinite dimensional linear equation for verifying almost everywhere stability of an invariant set in nonlinear systems.

The paper is organized as follows. In section II, we discuss some preliminaries from stochastic theory of dynamical system in particular the linear transfer operator (Perron-Frobenius) approach to the study of dynamical systems. We state the new definition of almost everywhere uniform stability of an invariant set and point out it difference to the almost everywhere notion of stability studied in our previous work [1]. In section III, we state and prove the main result of this paper giving necessary and sufficient condition for almost everywhere uniform stability of an invariant set. Conclusion follows in section IV.

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II. PRELIMINARIES AND DEFINITIONS

In this paper, *discrete dynamical systems* or mappings of the form

$$x_{n+1} = T(x_n) \tag{1}$$

are considered. $T : X \to X$ is in general assumed to be only continuous and non-singular with $X \subset \mathbb{R}^n$, a compact set. A mapping T is said to be non-singular with respect to a measure μ if $\mu(T^{-1}B) = 0$ for all $B \in \mathscr{B}(X)$ such that $\mu(B) = 0$. $\mathscr{B}(X)$ denotes the Borel σ-algebra on X, $\mathscr{M}(X)$ the vector space of real valued measures on $\mathscr{B}(X)$. Dynamical system (1) can be used to study the evolution of single trajectory, evolution of sets (measures supported on the sets) can be studied using linear transfer operators called as Perron-Frobenius operator and is defined as follows

Definition 1 (Perron-Frobenius operator): The Perron-Frobenius operator (P-F) $\mathbb{P} : \mathscr{M}(X) \to \mathscr{M}(X)$ corresponding to the dynamical system $T : X \to X$ is defined by

$$\mathbb{P}\mu(A) = \int_X \delta_{T(x)}(A) d\mu(x) = \int_X \chi_A d\mu(x) \tag{2}$$

where $\chi_A(\cdot)$ is the indicator function for the set A and $T^{-1}(A)$ is the pre-image set:

$$T^{-1}(A) := \{x \in X : T(x) \in A\}$$

We start with some measure theoretic preliminaries and definition of $\omega-$ limit set.

Definition 2 (Absolutely continuous measure): A measure μ is absolutely continuous with respect to another measure ϑ, denoted as $\mu \prec \vartheta$, if $\mu(B) = 0$ for all $B \in \mathscr{B}(X)$ with $\vartheta(B) = 0$.

Definition 3 (Equivalent measure): A measure μ is said to be equivalent to measure ϑ, denoted as $\mu \approx \vartheta$, provided $\mu(B) = 0$ if and only if $\vartheta(B) = 0$ for all $B \in \mathscr{B}(X)$.

Definition 4 (ω-limit set): A point $y \in X$ is called a ω- limit point for a point $x \in X$ if there exists a sequence of integers $\{n_k\}$ such that $T^{n_k}(x) \to y$ as $k \to \infty$. The set of all ω-limit point for x is denoted by $\omega(x)$ and is called its ω- limit set.

Definition 5 (Invariant set): A set $A \subset X$ is called *T-invariant* if

$$T(A) = A. \tag{3}$$

i.e., if $x \in A$ then $T^n(x) \in A$ for all $n \in \mathbb{Z}$.

A measure $\mu \neq 0$ is said to be a T-invariant measure if

$$\mu(B) = \mu(T^{-1}(B)) \tag{4}$$

for all $B \in \mathscr{B}(A)$. A T-invariant measure in Eq. (4) is a stochastic counterpart of the T-invariant set in Eq. (3) [4], [8]. For typical dynamical systems, the set A equals the support of *its* invariant measure μ.

In [1], global almost everywhere stability of an attractor set is studied, where the attractor set is defined as follows:

Definition 6 (Attractor set): A closed T invariant set $A \subset X$ is said to be an attractor set if it satisfies the following two properties

1) there exists a neighborhood $V \subset X$ of A such that $\omega(x)$ for almost every with respect to Lebesgue measure $x \in V$. V is called the local neighborhood of A.
2) there is no strictly smaller closed set $A^{'} \subset A$ which satisfies property 1.

So an attractor set is an invariant set which in addition has local attractive property. Following definition of almost everywhere stability is a global counterpart of the local attractive property on an invariant set.

Definition 7 (Almost everywhere stable): An invariant set A for the dynamical system $T : X \to X$ is said to be stable almost everywhere (a.e.) with respect to a finite measure $m \in \mathscr{M}(A^c)$ if

$$m\{x \in A^c : \omega(x) \not\subseteq A\} = 0$$

For the special case of a.e. stability of an equilibrium point x_0, the definition reduces to

$$m\{x \in X : \lim_{n\to\infty} T^n(x) \neq x_0\} = 0 \tag{5}$$

In our previous work [1], we proved a theorem giving sufficient condition for the almost everywhere stability of an attractor set. Lyapunov measure was introduced for verifying this weaker notion of almost everywhere stability. Above definition of almost everywhere stability does not impose any condition on the local stability of the invariant set. In fact the weak form of local stability of the invariant set is assumed by requiring that the invariant set is an attractor set. In this paper we introduce new definition of almost everywhere stability of an invariant set, we call it as *almost everywhere uniform stability*. The main result of this paper (section III) will give necessary and sufficient condition for verifying this stability definition. Before stating the definitions we introduce some notations. For any given $\delta > 0$, let U_δ be the δ neighborhood of an invariant set A and for any set $B \subset X \setminus U_\delta$, we denote by

$$B_n = \{x \in A^c : T^n(x) \in B\}$$

Definition 8: [Almost everywhere uniformly stable] An invariant set A for the dynamical system (1) is said to be almost everywhere uniformly stable w.r.t. measure m if for every $\varepsilon > 0$, there exists an $N(\delta, \varepsilon)$ such that

$$\sum_{n=N}^{\infty} m(B_n) < \varepsilon \tag{6}$$

for every set $B \subset X \setminus U_\delta$. Since $m(B_n) \geq 0$ for all n, the necessary condition for (6) to be true is that $\lim_{n\to\infty} m(B_n) = 0$, for every set $B \subset X \setminus U_\delta$.

So for a almost everywhere uniformly stable invariant set, the measure of the set of points that stays outside the δ neighborhood of an invariant set can be made arbitrarily small. In this paper we also prove necessary and sufficient condition for the almost everywhere stability of an attractor set with geometric decay. This definition of stability was introduced in [1] and is as follows:

Definition 9 (Almost everywhere stable with geometric decay): An invariant set A for dynamical system (1) is said to be stable almost everywhere with geometric decay w.r.t. measure m if there exists an $K(\delta) < \infty$ and $\beta < 1$ such that

$$m(B_n) < K\beta^n \tag{7}$$

for every set $B \subset A^c \setminus U(\delta)$.

A. Decomposition of P-F operator

Notice that the above definitions of stability of the invariant set is defined in terms of asymptotic behavior of the points in complement set A^c . This motivates us to define the restriction of the mapping $T : A^c \to X$. This restriction of T can be used to define a sub-stochastic Markov operator $\mathbb{P}_1$, which is defined as follows:

$$\mathbb{P}_1[\mu](B) := \int_{A^c} \delta_{T(x)}(B) d\mu(x) = \mu(T^{-1}B \cap A^c) \tag{8}$$

Sub Markov operator $\mathbb{P}_1$ is well-defined for $\mu \in \mathscr{M}(A^c)$ and $B \subset \mathscr{B}(A^c)$. Next, the restriction $T : A \to A$ can also be used to define a P-F operator denoted by

$$\mathbb{P}_0[\mu](B) = \int_B \delta_{T(x)}(B) d\mu(x), \tag{9}$$

where $\mu \in \mathscr{M}(A)$ and $B \subset \mathscr{B}(A)$.

The above considerations suggest a representation of the P-F operator $\mathbb{P}$ in terms of $\mathbb{P}_0$ and $\mathbb{P}_1$. This is indeed the case if one considers a splitting of the measure space

$$\mathscr{M}(X) = \mathscr{M}_0 \oplus \mathscr{M}_1, \tag{10}$$

where $\mathscr{M}_0 := \mathscr{M}(A)$ and $\mathscr{M}_1 := \mathscr{M}(A^c)$. Note that $\mathbb{P}_0 : \mathscr{M}_0 \to \mathscr{M}_0$ because $T : A \to A$ and $\mathbb{P}_1 : \mathscr{M}_1 \to \mathscr{M}_1$ by construction. It then follows that on the splitting defined by Eq. (10), the P-F operator has a lower-triangular matrix representation given by

$$\mathbb{P} = \begin{bmatrix} \mathbb{P}_0 & 0 \\ \times & \mathbb{P}_1 \end{bmatrix}. \quad (11)$$

For more details on the P-F decomposition refer [1].

III. Main result

In this section we present the main result of the paper giving necessary and sufficient condition for almost everywhere uniformly stability and almost everywhere stability with geometric decay of an invariant set in nonlinear systems.

Theorem 10: Consider the dynamical system $T : X \to X$, assumed to non-singular with respect to measure m with an invariant set $A \subset X$ and U_δ the δ neighborhood of the invariant set. The invariant set A is almost everywhere uniformly stable with respect to measure m if and only if there exists a measure $\bar{\mu}$ which is equivalent to measure m ($\bar{\mu} \approx m$) and is finite on $A^c \setminus U_\delta$ and satisfies

$$\mathbb{P}_1 \bar{\mu}(B) - \bar{\mu}(B) = -m(B) \quad (12)$$

for all sets $B \subset A^c \setminus U_\delta$.

Proof: Let the invariant set A be uniformly stable almost everywhere w.r.t. measure m. Construct the measure $\bar{\mu}$ as follows

$$\bar{\mu}(B) = \sum_{n=0}^{\infty} m(B_n) = \sum_{n=0}^{\infty} \mathbb{P}_1^n m(B) \quad (13)$$

where we have used the fact that

$$\mathbb{P}_1^n m(B) = m(T^{-n}(B) \cap A^c) = m\{x \in A^c : T^n(x) \in B\} = m(B_n)$$

Since the invariant set is almost everywhere uniformly stable (Def.(8)) the series on the right hand side of (13) converges for every set B by the Cauchy condition for series convergence [9]. $\bar{\mu}$ is finite on $A^c \setminus U_\delta$ because for any set $B \subset A^c \setminus U_\delta$ and every $\varepsilon > 0$, there exists an integer N such that

$$\bar{\mu}(B) = \sum_{k=0}^{N-1} m(B_n) + \sum_{k=N}^{\infty} m(B_n) < \sum_{k=0}^{N-1} m(B_n) + \varepsilon < K(\delta)$$

Since T is assumed to be non-singular with respect to measure m, the individual measure $\mathbb{P}_1^n m(B)$ are absolutely continuous with respect to measure m and hence $\bar{\mu} \prec m$. Measure m is absolutely continuous with respect to measure $\bar{\mu}$ follows from the construction of measure $\bar{\mu}$ and hence $\bar{\mu}$ is equivalent to measure m. Multiplying (13) on the both sides by $(\mathbb{P}_1 - I)$, we get

$$\mathbb{P}_1 \bar{\mu}(B) - \bar{\mu}(B) = \sum_{n=1}^{\infty} \mathbb{P}_1 m(B) - \sum_{n=0}^{\infty} \mathbb{P}_1 m(B) = -m(B)$$

thus verifying the claim (12) of the theorem.

To prove the necessary part assume that there exists a measure $\bar{\mu}$, such that $\bar{\mu} \approx m$ and is finite on $A^c \setminus U(\delta)$ and satisfies (12), then we have

$$(I - \mathbb{P}_1)\bar{\mu}(B) = m(B) \implies \bar{\mu}(B) = m(B) + \mathbb{P}_1 \bar{\mu}(B) \quad (14)$$

for any set $B \subset A^c \setminus U_\delta$. Multiplying both the sides of the above equality by $\mathbb{P}_1$, we get

$$\mathbb{P}_1 \bar{\mu}(B) = \mathbb{P}_1 m(B) + \mathbb{P}_1^2 \bar{\mu}(B) \quad (15)$$

Substituting (15) in (14) for $\mathbb{P}_1 \bar{\mu}$, we get

$$\bar{\mu}(B) = m(B) + \mathbb{P}_1 m(B) + \mathbb{P}_1^2 \bar{\mu}(B)$$

Hence by induction we have

$$\bar{\mu}(B) = \mathbb{P}_1^n \bar{\mu}(B) + \sum_{k=0}^{n} \mathbb{P}_1^k m(B)$$

Since $\mathbb{P}_1^n \bar{\mu}(B) \geq 0$ for all n and $\bar{\mu}$ is finite on $A^c \setminus U_\delta$ we get

$$\sum_{k=0}^{n} \mathbb{P}_1^k m(B) \leq \sum_{k=0}^{n} \mathbb{P}_1^k m(B) + \mathbb{P}_1^n \bar{\mu}(B) = \bar{\mu}(B) < K(\delta)$$

for all n. So for any given set B $\mathscr{P}_n := \sum_{k=0}^{n} \mathbb{P}_1^n m(B)$, is a increasing sequence of real numbers which is bounded from above and and hence from Cauchy condition of series convergence converges which then implies almost everywhere uniform stability of the invariant set A. ■

Theorem (13) is similar to the theorem on necessary and sufficient condition for asymptotically stability of equilibrium point in linear systems [3]. The difference here is that the algebraic Lyapunov equation for verifying asymptotically stability in linear system is replaced by the infinite dimensional linear equation (12) for verifying almost everywhere stability in nonlinear system. Equation (12) will be referred to as *Lyapunov measure equation* and the positive solution $\bar{\mu}$ if it exists as *Lyapunov measure*. In theorem (13), the non-singular measure m, which is of particular interest is the Lebesgue measure. Another measure which is of particular interest can be constructed as follows:

$$m_S(\cdot) = m(S \cap \cdot)$$

where m in this case is the Lebesgue measure, so m_S is the Lebesgue measure supported on set S. Measure m_S is not necessarily a non-singular measure, in particular when the set $T(S)$ is disjoint from S, then $m_S \circ T^{-1}$ is not absolutely continuous with respect to measure m_S. This measure m_S can however be used to characterize the domain of attraction or to study the local stability of an invariant set A. In particular one might be interested in almost everywhere stability of an invariant set A with respect to initial conditions starting from the set S. This can be defined as follows:

Definition 11: Invariant set A for the dynamical system $T : X \to X$ is said to be Lebesgue almost everywhere uniformly stable with respect to initial conditions starting from the set $S \subset X \setminus U_\delta$ if for any given $\varepsilon > 0$, there exists an $N(\varepsilon) < \infty$ such that

$$\sum_{n=N}^{\infty} m\{x \in S : T^n(x) \in B\} < \varepsilon$$

for every set $B \subset X \setminus U_\delta$, where m in this case is the Lebesgue measure.

We have following theorem for the necessary and sufficient condition for the almost everywhere uniform stability of an invariant set with respect to almost every initial condition starting from the set S.

Theorem 12: The invariant set A for the dynamical system $T : X \to X$ is almost everywhere uniformly stable with respect to initial conditions starting from the set S if and only if there exists a measure $\bar{\mu}_S$ which is finite on $X \setminus U_\delta$ such that $m_S \prec \bar{\mu}_S$ and satisfies

$$\mathbb{P}_1 \bar{\mu}_S(B) - \bar{\mu}_S(B) = -m_S(B) \tag{16}$$

for every set $B \subset X \setminus U_\delta$, where m_S is the lebesgue measure supported on set S.

The proof of the above theorem will follow exactly along the same line as the proof of the main theorem (13). The only difference been the measure m_S will be absolutely continuous with respect to $\bar{\mu}$.

Now we prove the theorem for necessary and sufficient condition for almost everywhere stability with geometric decay.

Theorem 13: Consider the dynamical system $T : X \to X$, assumed to be non-singular with respect to measure m with an invariant set $A \subset X$ and U_δ the δ neighborhood of the invariant set. The invariant set A is almost everywhere stable with geometric decay with respect to measure m if and only if there exists a measure $\bar{\mu}$ which is equivalent to measure m ($\bar{\mu} \approx m$) and is finite on $A^c \setminus U_\delta$ and satisfies

$$\alpha \mathbb{P}_1 \bar{\mu}(B) - \bar{\mu}(B) = -m(B) \tag{17}$$

for some $\alpha > 1$ and every set $B \subset A^c \setminus U_\delta$.

Proof: Assume that the invariant set A is stable almost everywhere with geometric decay then we know that

$$m(B_n) = \mathbb{P}_1^n m(B) < K\beta^n \tag{18}$$

Let $\beta = \beta_1\beta_2$, where $\beta_i < 1$ for $i = 1,2$, so we have

$$\alpha^n \mathbb{P}_1^n m(B) < K\beta_2^n \tag{19}$$

where $\alpha = \beta_1^{-1} > 1$. Construct the measure $\bar{\mu}$ as follows

$$\bar{\mu}(B) = \sum_{k=0}^{\infty} \alpha^k \mathbb{P}_1^k m(B) < K \sum_{k=0}^{\infty} \beta_2^k = \frac{K}{1-\beta_2} \tag{20}$$

This shows that measure $\bar{\mu}$ is finite on $A^c \setminus U(\delta)$. Measure $\bar{\mu}$ is equivalent to measure m follows by exactly using the same argument as in theorem (13). Now applying $\alpha\mathbb{P}_1 - I$ on both the sides of (20), we get

$$(\alpha\mathbb{P}_1 - I)\bar{\mu}(B) = -m(B) \tag{21}$$

Thus verifying the claim. To prove the necessary part, assume that there exists a measure $\bar{\mu}$ which is finite on $A^c \setminus U_\delta$ and satisfies

$$(\alpha\mathbb{P}_1 - I)\bar{\mu}(B) = -m(B) \implies \bar{\mu}(B) = m(B) + \alpha\mathbb{P}_1\bar{\mu}(B) \tag{22}$$

for some $\alpha > 1$. Applying $\alpha^n\mathbb{P}_1$ on both the sides of above expression and after simplification, we get

$$\bar{\mu}(B) = \alpha^n \mathbb{P}_1 \bar{\mu}(B) + \sum_{k=0}^{n} \alpha^n \mathbb{P}_1^n m(B)$$

Since $\bar{\mu}$ is finite on $A^c \setminus U_\delta$, we have

$$\sum_{k=0}^{n} \alpha^n \mathbb{P}_1^n m(B) < \alpha^n \mathbb{P}_1 \bar{\mu}(B) + \sum_{k=0}^{n} \alpha^n \mathbb{P}_1^n m(B) = \bar{\mu}(B) < K(\delta)$$

for all n and hence

$$\mathbb{P}_1^n m(B) < \alpha^{-n} K(\delta) = \beta^n K(\delta)$$

for some $\beta < 1$ and hence stability with geometric decay. ■

In addition to verifying stability and characterizing the domain of attraction, Lyapunov measure equation and Lyapunov measure can also be used to characterize the region of the phase space where system trajectories do not enter. Characterization of such region using Lyapunov measure will have important application for motion planning problem, where one is interested in designing system trajectory to go from one region of the phase space to another while avoiding some obstacle set in the phase space. We begin with the definition of this problem.

Definition 14 (A.e asymptotic steering avoiding obstacle): The dynamical system $T : X \to X$ is said to steer almost every with respect to Lebesgue measure initial states starting from the initial set $S_i \subset X \setminus U_\delta$ to the invariant set A uniformly while avoiding the obstacle set $S_o \subset X \setminus U_\delta$ if

1) For every $\varepsilon > 0$ there exists $N(\varepsilon) < \infty$ such that

$$\sum_{n=N}^{\infty} m(B_n^{'}) < \varepsilon, \qquad ; B_n^{'} = \{x \in S_i : T^n(x) \in B\}$$

for every set $B \subset X \setminus U_\delta$ and U_δ is the δ neighborhood of the invariant set A.

2) For

$$S_o^n := \{x \in S_i : T^n(x) \in S_o\}, \qquad m(S_o^n) = 0 \quad \forall n > 0$$

where m in this case is assumed to be Lebesgue measure and $S_i \cap S_o = \emptyset$

So the condition 1. of this definition guarantee that almost every with respect to Lebesgue measure initial condition starting from the initial set S_i will enter the δ neighborhood of an invariant set A and condition 2. ensures that the set of points that enter inside the obstacle set S_o at any time $n > 0$ is a measure zero set. Now we state and prove a theorem which gives necessary and sufficient condition in terms of Lyapunov measure for a.e. asymptotic steering avoiding obstacle set.

Theorem 15: The dynamical system $T : X \to X$ will steer almost every initial condition starting from the initial set S_i to the invariant set A uniformly while avoiding the obstacle set S_o (Def. 14) if and only if there exists a measure $\bar{\mu}$, which is finite on $X \setminus U_\delta$ and satisfies following conditions

$$\mathbb{P}_1 \bar{\mu}(B) - \bar{\mu}(B) = -m_{S_i}(B) \tag{23}$$

for every set $B \subset X \setminus U_\delta$ and

$$\bar{\mu}(S_o) = 0 \tag{24}$$

where m_{S_i} is the Lebesgue measure supported on the initial set S_i.

Proof: Assume that there exists a measure $\bar{\mu}$ finite on $X \setminus U_\delta$, then by (23), we have

$$\bar{\mu}(B) = \mathbb{P}_1 \bar{\mu}(B) + m_{S_i}(B) \tag{25}$$

Multiplying the above equation from both the sides by $\mathbb{P}_1$, we get

$$\mathbb{P}_1 \bar{\mu}(B) = \mathbb{P}_1^2 \bar{\mu}(B) + \mathbb{P}_1 m_{S_i}(B)$$

Substituting for $\mathbb{P}_1 \bar{\mu}(B)$ in (25), we get

$$\bar{\mu}(B) = \mathbb{P}_1^2 \bar{\mu}(B) + \mathbb{P}_1 m(B) + m_{S_i}(B)$$

by induction, we have

$$\bar{\mu}(B) = \mathbb{P}_1^n \bar{\mu}(B) + \sum_{k=0}^{n} \mathbb{P}_1^k m_{S_i}(B)$$

Since $\bar{\mu}(B)$ is finite on $X \setminus U_\delta$ and $\mathbb{P}_1^n \bar{\mu}(B) \geq 0$ for all n, we have

$$\sum_{k=0}^{n} \mathbb{P}_1^k m_{S_i}(B) < \mathbb{P}_1^n \bar{\mu}(B) + \sum_{k=0}^{n} \mathbb{P}_1^k m_{S_i}(B) = \bar{\mu}(B) < K(\delta) < \infty$$

Since this is true for all n and the individual measure $\mathbb{P}_1^n m_{S_i}(B) \geq 0$, we have for any given $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that

$$\sum_{k=N}^{n} \mathbb{P}_1^k m_{S_i}(B) < \varepsilon$$

The almost everywhere asymptotic steering from set S_i to the invariant set A follows by using the fact that

$$\mathbb{P}_1^n m_{S_i}(B) = m_{S_i}\{x \in A^c : T^n(x) \in B\} = m\{x \in S_i : T^n(x) \in B\}$$

since $S_i \subset X \setminus U_\delta \subset A^c$. To prove that almost every trajectory do not intersect the obstacle set S_o, consider that opposite is true i.e.,

$$m\{x \in S_i : T^n(x) \in S_o\} > 0 \implies \mathbb{P}_1^n m_{S_i}(S_o) > 0 \tag{26}$$

for some $n > 0$. In the first part of the proof we have proved that the almost every w.r.t. measure m initial condition starting from the set S_i will converge to the invariant set A. This allows us to write the solution $\bar{\mu}$ for equation (23) as a infinite series involving iterates of $\mathbb{P}_1^n$ as follows:

$$\bar{\mu}(B) = \sum_{n=0}^{\infty} \mathbb{P}_1^n m_{S_i}(B)$$

where $\bar{\mu}$ is finite for all $B \subset X \setminus U_\delta$ and hence in particular for $B = S_o$, we have from (26)

$$\bar{\mu}(S_o) = \sum_{n=1}^{\infty} \mathbb{P}_1^n m_{S_i}(S_o) > 0$$

where we have use the fact that $m_{S_i}(S_o) = 0$ and (26). This gives us contradiction to the equation (24) in the theorem. To prove other way around construct measure $\bar{\mu}$ as follows

$$\bar{\mu}(B) = \sum_{n=0}^{\infty} \mathbb{P}_1^n m_{S_i}(B) \tag{27}$$

The measure $\bar{\mu}$ constructed above is finite on $X \setminus U_\delta$ because the invariant set A is assumed to be a.e. uniformly stable with respect to initial conditions starting from the set S_i. Applying $\mathbb{P}_1 - I$ on both the sides of (27), we get

$$\mathbb{P}_1 \bar{\mu}(B) - \bar{\mu}(B) = -m_{S_i}(B)$$

thus verifying the claim (23) of the theorem. Now we show that for the $\bar{\mu}$ constructed above $\bar{\mu}(S_o) = 0$. This follows because of the fact that the set of trajectories starting from the initial conditions from set S_i and entering the obstacle set S_o has Lebesgue measure zero. Hence we have

$$S_o^n := \{x \in S_i : T^n(x) \in S_o\} = 0 \quad \forall n > 0$$

which implies

$$\mathbb{P}_1^n m_{S_i}(S_o) = 0 \quad \forall n > 0$$

which together with the fact that $S_i \cap S_o = \emptyset$, gives us

$$\bar{\mu}(S_o) = 0$$

■

IV. Conclusion and Discussion

Result on necessary and sufficient condition for the almost everywhere uniform stability of an invariant set is presented. Condition is presented in terms of positive solution (Lyapunov measure) of Lyapunov measure equation. Lyapunov measure equation thus form the counterpart of finite dimensional matrix Lyapunov equation, and is used for verifying weaker notion of almost everywhere stability. Finite dimensional approximation of the Lyapunov measure equation and its solution can be obtained using set oriented numerical methods. Preliminary work in that direction is already presented in [2]. Lyapunov function and Lyapunov equation play a very important role in linear control theory, we believe that the Lyapunov measure equation and Lyapunov measure will also play a similar important role for nonlinear systems.

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