Optimal stabilization using Lyapunov measures

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Abstract

Numerical solutions for the optimal feedback stabilization of discrete time dynamical systems is the focus of this paper. Set-theoretic notion of almost everywhere stability introduced by the Lyapunov measure, which is weaker than conventional Lyapunov function-based stabilization methods, is used for optimal stabilization. The linear Perron-Frobenius transfer operator is used to pose the optimal stabilization problem as an infinite dimensional linear program. Set-oriented numerical methods are used to obtain the finite dimensional approximation of the linear program. We provide conditions for the existence of stabilizing feedback controls and show the optimal stabilizing feedback control can be obtained as a solution of a finite dimensional linear program. The approach is demonstrated on stabilization of period two orbit in controlled standard map.

I. Introduction

Stability analysis and stabilization of nonlinear systems are two of the most important and extensively studied problems in control theory. Lyapunov functions are used for stability analysis and control Lyapunov functions (CLF) are used in the design of stabilizing feedback controllers. Under the assumption of detectability and stabilizability of the nonlinear system, a positive valued optimal cost function of an optimal control problem (OCP) can also be used as a control Lyapunov function. The optimal controls of OCP are obtained as the solution of the Hamilton Jacobi Bellman (HJB) equation. The HJB equation is a nonlinear partial differential equation and one must resort to approximate numerical schemes for its solution. Numerical schemes typically discretize the

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state-space; hence, the resulting problem size grows exponentially with the dimension of the state-space. This is commonly referred to as the *curse of dimensionality*. The approach is particularly attractive for feedback control of nonlinear systems with lower dimensional state space. The method proposed in this paper also suffers from the same drawback. Among the vast literature available on the topic of solution of HJB equation, we briefly review some of the related literature.

Vinter [1] was the first to propose a linear programming approach for nonlinear optimal control of continuous time systems. This was exploited to develop a numerical algorithm, based on semidefinite programming and density function-based formulation by Rantzer and co-workers in [2], [3]. Global almost everywhere stability of stochastic systems was studied by van Handel [4] using the density function. Lasserre, Hernández-Lerma, and co-workers [5], [6] formulated the control of Markov processes as a solution of the HJB equation. An adaptive space discretization approach is used in [7]; a cell mapping approach is used in [8] and [9], [10] utilize set oriented numerical methods to convert the HJB to one of finding the minimum cost path on a graph derived from transition. In [11], [12], [13], solutions to stochastic and deterministic optimal control problems are proposed using a linear programming approach or using a sequence of LMI relaxations. Our paper also draws some connection to research on optimization and stabilization of controlled Markov chains discussed in [14]. Computational techniques based on the viscosity solution of HJB equation is proposed for the approximation of value function and optimal controls in [15] (Chapter VI).

Our proposed method, in particular the computational approach, draw some similarity with the above discussed references on the approximation of the solution of HJB equation [8], [9], [10], [15]. In that our method too relies on discretization of state space to obtain globally optimal stabilizing control. However, our proposed approach differs from the above references in the following two fundamental ways. The first main difference arise due to adoption of non-classical weaker set-theoretic notion of almost everywhere stability for optimal stabilization. This weaker notion of stability allows for the existence of unstable dynamics in the complement of the stabilized attractor set whereas such unstable dynamics are not allowed using the classical notion of Lyapunov stability adopted in the above references. This weaker notion of stability is advantageous from the point of view of feedback control design. The notion of almost everywhere stability and density function for its verification was introduced by Rantzer in [16]. Furthermore, the author in [16] proved that unlike control Lyapunov function, the co-design problem of jointly finding the density function and the stabilizing controller is convex [17]. The Lyapunov measure used in this paper for optimal stabilization can be viewed as a measure corresponding to the density function [18], [19] and hence it enjoys the same convexity property for the controller design. This convexity property combined with the proposed linear transfer operator framework is precisely exploited in the development of linear programming-based computational framework for optimal stabilization using Lyapunov measures. The second main difference compared to references [14], [15] is in the use of discount factor $\gamma > 1$ in the cost function (refer to Remark 11). We show in section V the discount factor play an important role in controlling the effect of finite dimensional discretization or approximation process on the true solution. In particular, we show that by allowing for the discount factor γ to be greater than one it is possible to ensure that the control obtained using the finite dimensional approximation is *truly* stabilizing the nonlinear system. For more details on the comparison of our proposed approach with optimal control via approximate solutions of HJB equation, we refer the readers to [20] (Section IV E) involving second author.

In a previous work [20] involving the second author, the problem of designing deterministic feedback controllers for stabilization via control Lyapunov measure was addressed. The authors proposed solving the problem by using a mixed integer formulation or a non-convex nonlinear program, which are not computationally efficient. There are two main contributions of this paper. First, we show a deterministic stabilizing feedback controller can be constructed using a computationally cheap tree-growing algorithm (Algorithm 1, Lemma 13). The second main contribution of this paper is the extension of the Lyapunov measure framework introduced in [18] to the design of optimal stabilization of an attractor set. We prove the optimal stabilizing controllers can be obtained as the solution to a linear program. Unlike the approach proposed in [20], the solution to the linear program is guaranteed to yield deterministic controls. This paper is an extended version of the paper that appeared in the 2008 American Control Conference [21].

This paper is organized as follows. In Section II, we provide a brief overview of key results from [18], [20] for stability analysis and stabilization of nonlinear systems using the Lyapunov measure. The transfer operators-based framework is used to formulate the OCP as an infinite dimensional linear program in Section III. A computational approach, based on set-oriented numerical methods, is proposed for the finite dimensional approximation of the linear program in Section IV. The effect of discretization on the optimal solution is discussed in Section V followed by simulation results in Section VI, and conclusions in Section VII.

II. Lyapunov measure, stability and stabilization

The *Lyapunov measure* and *control Lyapunov measure* were introduced in [20], [18], for stability analysis and stabilizing controller design in discrete-time dynamical systems of the form,

$$x_{n+1} = F(x_n), \tag{1}$$

where $F : X \to X$ is assumed to be continuous with $X \subset \mathbb{R}^q$, a compact set. We denote by $\mathcal{B}(X)$ the Borel-σ algebra on X and $\mathcal{M}(X)$, the vector space of real valued measure on $\mathcal{B}(X)$. The mapping F is assumed to be nonsingular with respect to the Lebesgue measure ℓ, i.e., $\ell(T^{-1}(B)) = 0$, for all sets $B \in \mathcal{B}(X)$, such that $\ell(B) = 0$. In this paper, we are interested in optimal stabilization of an attractor set defined as follows:

Definition 1 (Attractor set): A set $\mathcal{A} \subset X$ is said to be forward invariant under F if $F(\mathcal{A}) = \mathcal{A}$. A closed forward invariant set $\mathcal{A}$ is said to be an attractor set, if it there exists a neighborhood $V \subset X$ of $\mathcal{A}$, such that $\omega(x) \subset \mathcal{A}$ for all $x \in V$, where $\omega(x)$ is the ω limit set of x [18].

Remark 2: In the following definitions and theorems, we will use the notation, $U(\epsilon)$, to denote the $\epsilon > 0$ neighborhood of the attractor set $\mathcal{A}$ and $m \in \mathcal{M}(X)$, a finite measure absolutely continuous with respect to Lebesgue.

Definition 3 (Almost everywhere stable with geometric decay): The attractor set $\mathcal{A} \subset X$ for a dynamical system (1) is said to be almost everywhere (a.e.) stable with geometric decay with respect to some finite measure, $m \in \mathcal{M}(X)$, if given any $\epsilon > 0$, there exists $M(\epsilon) < \infty$ and $\beta < 1$, such that $m\{x \in \mathcal{A}^c : F^n(x) \in X \setminus U(\epsilon)\} < M(\epsilon)\beta^n$.

The above set-theoretic notion of a.e. stability is introduced in [18] and verified using the linear transfer operator framework. For the discrete time dynamical system (1), the linear transfer Perron Frobenius (P-F) operator [22] denoted by $\mathbb{P}_F : \mathcal{M}(X) \to \mathcal{M}(X)$ is given by,

$$[\mathbb{P}_F\mu](B) = \int_X \chi_B(F(x))d\mu(x) = \mu(F^{-1}(B)), \tag{2}$$

where $\chi_B(x)$ is the indicator function supported on the set $B \in \mathcal{B}(X)$ and $F^{-1}(B)$ is the inverse image of set B. We define a sub-stochastic operator as a restriction of the P-F operator on the complement of the attractor set as follows:

$$[\mathbb{P}^1_F\mu](B) := \int_{\mathcal{A}^c} \chi_B(F(x))d\mu(x), \tag{3}$$

for any set $B \in \mathcal{B}(\mathcal{A}^c)$ and $\mu \in \mathcal{M}(\mathcal{A}^c)$. The condition for the a.e. stability of an attractor set $\mathcal{A}$ with respect to some finite measure m is defined in terms of the existence of the *Lyapunov measure* $\bar{\mu}$, defined as follows [18]:

Definition 4 (Lyapunov measure): The Lyapunov measure is defined as any non-negative measure $\bar{\mu}$, finite outside $U(\epsilon)$ (see Remark 2), and satisfies the following inequality, $[\mathbb{P}^1_F\bar{\mu}](B) < \gamma^{-1}\bar{\mu}(B)$, for some $\gamma \geq 1$ and all sets $B \in \mathcal{B}(X \setminus U(\epsilon))$, such that $m(B) > 0$.

The following theorem from [23] provides the condition for a.e. stability with geometric decay.

Theorem 5: An attractor set $\mathcal{A}$ for the dynamical system (1) is a.e. stable with geometric decay with respect to finite measure m, if and only if for all $\epsilon > 0$ there exists a non-negative measure $\bar{\mu}$ which is finite on $\mathcal{B}(X \setminus U(\epsilon))$ and satisfies

$$\gamma[\mathbb{P}^1_F\bar{\mu}](B) - \bar{\mu}(B) = -m(B) \tag{4}$$

for all measurable sets $B \subset X \setminus U(\epsilon)$ and for some $\gamma > 1$.

Proof 6:

We first prove the necessary part. Let the attractor set $\mathcal{A}$ be a.e. geometric stable w.r.t. finite measure m, then by definition 3 we know that

$$m\{x \in \mathcal{A}^c : F^n(x) \in B\} < M(\epsilon)\beta^n$$

for some $0 < \beta < 1$. For any measurable set $B \subset X \setminus U(\epsilon)$ and using (3), we have $[\mathbb{P}_F^1 m](B) = m(F^{-1}(B) \cap \mathcal{A}^c)$. Furthermore since $\mathcal{A}$ is forward invariant, we have $F^{-1}(\mathcal{A}^c) \subset \mathcal{A}^c$ and hence $[\mathbb{P}_F^1 m]^2(B) = m(F^{-1}(F^{-1}(B) \cap \mathcal{A}^c) \cap \mathcal{A}^c) = m(F^{-2}(B) \cap \mathcal{A}^c)$. Using induction, we get

$$[\mathbb{P}_F^1 m]^n(B) = m(F^{-n}(B) \cap \mathcal{A}^c) = m\{x \in \mathcal{A}^c : F^n(x) \in B\} \tag{5}$$

Let $\beta = \beta_1\beta_2$, where $0 < \beta_i < 1$ for $i = 1, 2$. So we have $\gamma^n[\mathbb{P}_F^1 m]^n(B) < M(\epsilon)\beta_2^n$, where $\gamma = \frac{1}{\beta_1} > 1$. Consider the following construction for the measure $\bar{\mu}$

$$\bar{\mu}(B) = \sum_{k=0}^{\infty} \gamma^k [\mathbb{P}_F^1 m]^k(B) < M(\epsilon) \sum_{k=0}^{\infty} \beta_2^k = \frac{M(\epsilon)}{1 - \beta_2}$$

This proves that $\bar{\mu}$ is finite on all sets $B \subset X \setminus U(\epsilon)$. The above construction of $\bar{\mu}$ can be shown to satisfy (4) after multiplying the above equality by $\gamma\mathbb{P}_F^1 - I$ on both the sides and after simplification. To prove the sufficiency part, we assume that there exists a Lyapunov measure $\bar{\mu}$ which is finite on $\mathcal{B}(X \setminus U(\epsilon))$ and satisfies (4). From (4), we write $\bar{\mu}(B) = \gamma[\mathbb{P}_F^1 \bar{\mu}](B) + m(B)$. Multiplying both sides of the above equation by $\gamma\mathbb{P}_F^1$ and after simplification we get

$$\bar{\mu}(B) = \gamma^2[\mathbb{P}_F^1 \bar{\mu}]^2(B) + \gamma[\mathbb{P}_F^1 m](B) + m(B)$$

The above process of multiplying on both the sides by $\gamma\mathbb{P}_F^1$ can be continued recursively to get

$$\bar{\mu}(B) = \gamma^n [\mathbb{P}_F^1 \bar{\mu}]^n(B) + \sum_{k=0}^{n-1} \gamma^k [\mathbb{P}_F^1 m]^k(B)$$

Since $\bar{\mu}$ is finite on $\mathcal{B}(X \setminus U(\epsilon))$, we have $\sum_{k=0}^{n-1} \gamma^k [\mathbb{P}_F^1 m]^k(B) < \bar{\mu}(B) < M(\epsilon)$, for all n. Using (5), we get $m\{x \in \mathcal{A}^c : T^k(x) \in B\} = [\mathbb{P}_F^1 m]^k(B) < M(\epsilon)\gamma^{-k} = M(\epsilon)\beta^k$ where $\beta < 1$. This proves a.e. stability of $\mathcal{A}$ with geometric decay using (5) .

We consider the stabilization of dynamical systems of the form $x_{n+1} = T(x_n, u_n)$, where $x_n \in X \subset \mathbb{R}^q$ and $u_n \in U \subset \mathbb{R}^d$ are the state and the control input, respectively. Both X and U are assumed compact. The objective is to design a feedback controller, $u_n = K(x_n)$, to stabilize the attractor set $\mathcal{A}$. The stabilization problem is solved using the Lyapunov measure by extending the P-F operator formalism to the control dynamical system [20]. We define the feedback control mapping $C : X \to Y := X \times U$ as $C(x) = (x, K(x))$. We denote by $\mathcal{B}(Y)$ the Borel-σ algebra on Y and $\mathcal{M}(Y)$ the vector space of real valued measures on $\mathcal{B}(Y)$. For any $\mu \in \mathcal{M}(X)$, the control mapping C can be used to define a measure, $\theta \in \mathcal{M}(Y)$, as follows:

$$\theta(D) := [\mathbb{P}_C \mu](D) = \mu(C^{-1}(D)), \quad [\mathbb{P}_{C^{-1}}\theta](B) := \mu(B) = \theta(C(B)), \tag{6}$$

for all sets $D \in \mathcal{B}(Y)$ and $B \in \mathcal{B}(X)$. Since C is an injective function with θ satisfying (6), it follows from the theorem on disintegration of measure [24] (Theorem 5.8) there exists a unique disintegration θ_x of the measure θ for μ almost all $x \in X$, such that $\int_Y f(y)d\theta(y) = \int_X \int_{C(x)} f(y)d\theta_x(y)d\mu(x)$, for any Borel-measurable function $f : Y \to \mathbb{R}$. In particular, for $f(y) = \chi_D(y)$, the indicator function for the set D, we get $\theta(D) = \int_X \int_{C(x)} \chi_D(y)d\theta_x(y)d\mu(x) = [\mathbb{P}_C\mu](D)$. Using the definition of the feedback controller mapping C, we write the feedback control system as $x_{n+1} = T(x_n, K(x_n)) = T \circ C(x_n)$. The system mapping $T : Y \to X$ can be associated with P-F operators $\mathbb{P}_T : \mathcal{M}(Y) \to \mathcal{M}(X)$ as $[\mathbb{P}_T\theta](B) = \int_Y \chi_B(T(y))d\theta(y)$. The P-F operator for the composition $T \circ C : X \to X$ can be written as a product of $\mathbb{P}_T$ and $\mathbb{P}_C$. In particular, we obtain

$$[\mathbb{P}_{T\circ C}\mu](B) = \int_Y \chi_B(T(y))d[\mathbb{P}_C\mu](y) = [\mathbb{P}_T\mathbb{P}_C\mu](B) = \int_X \int_{C(x)} \chi_B(T(y))d\theta_x(y)d\mu(x).$$

The P-F operators, $\mathbb{P}_T$ and $\mathbb{P}_{\mathbb{C}}$, are used to define their restriction, $\mathbb{P}^1_T : \mathcal{M}(\mathcal{A}^c \times U) \to \mathcal{M}(\mathcal{A}^c)$, and $\mathbb{P}^1_C : \mathcal{M}(\mathcal{A}^c) \to \mathcal{M}(\mathcal{A}^c \times U)$ to the complement of the attractor set, respectively, in a way similar to the Eq. (3). The control Lyapunov measure introduced in [20] is defined as any non-negative measure $\bar{\mu} \in \mathcal{M}(\mathcal{A}^c)$, finite on $\mathcal{B}(X \setminus U(\epsilon))$, such that there exists a control mapping C that satisfies $[\mathbb{P}^1_T\mathbb{P}^1_C\bar{\mu}](B) < \beta\bar{\mu}(B)$, for every set $B \in \mathcal{B}(X \setminus U(\epsilon))$ and $\beta \leq 1$. Stabilization of the attractor set is posed as a co-design problem of jointly obtaining the control Lyapunov measure $\bar{\mu}$ and the control P-F operator $\mathbb{P}_C$ or in particular disintegration of measure θ, i.e., θ_x. The disintegration measure θ_x, which lives on the fiber of $C(x)$, in general, will not be absolutely continuous with respect to Lebesgue. For the deterministic control map, $K(x)$, the conditional measure, $\theta_x(u) = \delta(u - K(x))$, the Dirac delta measure. However, for the purpose of computation, we relax this condition. The purpose of this paper and the following sections are to extend the Lyapunov measure-based framework for the optimal stabilization of nonlinear systems. One of the key highlights of this paper is the *deterministic* finite optimal stabilizing control is obtained as the solution for a finite linear program.

III. Optimal stabilization

The objective is to design a feedback controller for the stabilization of the attractor set, $\mathcal{A}$, in a.e. sense, while minimizing a suitable cost function. Consider the following control system,

$$x_{n+1} = T(x_n, u_n), \tag{7}$$

where $x_n \in X \subset \mathbb{R}^q$ and $u_n \in U \subset \mathbb{R}^d$ are state and control input, respectively, and $T : X \times U =: Y \to X$. Both X and U are assumed compact. We define $X_1 := X \setminus U(\epsilon)$.

Assumption 7: We assume there exists a feedback controller mapping $C_0(x) = (x, K_0(x))$, which locally stabilizes the invariant set $\mathcal{A}$, i.e., there exists a neighborhood V of $\mathcal{A}$ such that $T \circ C_0(V) \subset V$ and $x_n \to \mathcal{A}$ for all $x_0 \in V$; moreover $\mathcal{A} \subset U(\epsilon) \subset V$.

Our objective is to construct the optimal stabilizing controller for almost every initial condition starting from X_1. Let $C_1 : X_1 \to Y$ be the stabilizing control map for X_1. The control mapping $C : X \to X \times U$ can be written as follows:

$$C(x) = \begin{cases} C_0(x) = (x, K_0(x)) & \text{for} \quad x \in U(\epsilon) \\ C_1(x) = (x, K_1(x)) & \text{for} \quad x \in X_1. \end{cases} \tag{8}$$

Furthermore, we assume the feedback control system $T \circ C : X \to X$ is non-singular with respect to the Lebesgue measure, m. We seek to design the controller mapping, $C(x) = (x, K(x))$, such that the attractor set $\mathcal{A}$ is a.e. stable with geometric decay rate $\beta < 1$, while minimizing the following cost function,

$$\mathcal{C}_C(B) = \int_B \sum_{n=0}^{\infty} \gamma^n G \circ C(x_n) dm(x), \tag{9}$$

where $x_0 = x$, the cost function $G : Y \to \mathbb{R}$ is assumed a continuous non-negative real-valued function, such that $G(\mathcal{A}, 0) = 0$, $x_{n+1} = T \circ C(x_n)$, and $0 < \gamma < \frac{1}{\beta}$. Note, that in the cost function

(9), γ is allowed greater than one and this is one of the main departures from the conventional optimal control problem, where $\gamma \leq 1$. However, under the assumption that the controller mapping C renders the attractor set a.e. stable with a geometric decay rate, $\beta < \frac{1}{\gamma}$, the cost function (9) is finite.

Remark 8: To simplify the notation, in the following we will use the notion of scalar product between continuous function $h \in \mathcal{C}^0(X)$ and measure $\mu \in \mathcal{M}(X)$ [22] $\langle h, \mu \rangle_X := \int_X h(x) d\mu(x)$. The following theorem proves the cost of stabilization of the set $\mathcal{A}$ as given in Eq. (9) can be expressed using the control Lyapunov measure equation.

Theorem 9: Let the controller mapping, $C(x) = (x, K(x))$, be such that the attractor set $\mathcal{A}$ for the feedback control system $T \circ C : X \to X$ is a.e. stable with geometric decay rate $\beta < 1$. Then, the cost function (9) is well defined for $\gamma < \frac{1}{\beta}$ and, furthermore, the cost of stabilization of the attractor set $\mathcal{A}$ with respect to Lebesgue almost every initial conditions starting from set $B \in \mathcal{B}(X_1)$ can be expressed as follows:

$$\mathcal{C}_C(B) = \int_B \sum_{n=0}^{\infty} \gamma^n G \circ C(x) dm(x) = \int_{\mathcal{A}^c \times U} G(y) d[\mathbb{P}_C^1 \bar{\mu}_B](y) = \left\langle G, \mathbb{P}_C^1 \bar{\mu}_B \right\rangle_{\mathcal{A}^c \times U}, \tag{10}$$

where $\bar{\mu}_B$ is the solution of the following control Lyapunov measure equation,

$$\gamma \mathbb{P}_T^1 \cdot \mathbb{P}_C^1 \bar{\mu}_B(D) - \bar{\mu}_B(D) = -m_B(D), \text{ for every set } D \in \mathcal{B}(X_1), \tag{11}$$

where $m_B(\cdot) := m(B \cap \cdot)$ is a finite measure supported on the set $B \in \mathcal{B}(X_1)$.

Proof 10: The controller mapping $C : X \to Y$ is assumed to be stabilizing the attractor set $\mathcal{A}$ a.e. with geometric decay rate $\beta < \frac{1}{\gamma}$. Hence there exists an $M(\epsilon) < \infty$ such that

$$m\{x \in \mathcal{A}^c : (T \circ C)^n(x) \in D\} < M(\epsilon)\beta^n$$

for every set $D \subset X_1 := X \setminus U(\epsilon)$. Now using the result from Theorem 5, we know that there exists a non-negative measure $\bar{\mu}$ which is finite on $\mathcal{B}(X_1)$ and satisfies

$$\gamma[\mathbb{P}_{T \circ C}^1 \bar{\mu}](D) - \bar{\mu}(D) = -m(D).$$

From (II) we have that

$$\gamma[\mathbb{P}_T^1 \cdot \mathbb{P}_C^1 \bar{\mu}](D) - \bar{\mu}(D) = -m(D)$$

For the cost of stabilization of a set B, we have

$$\mathcal{C}_C(B)=\int_B\sum_{n=0}^{\infty}\gamma^n G\circ C(x_n)dm(x)=\int_B\lim_{N\to\infty}\sum_{n=0}^{N}\gamma^n G\circ C(x_n)dm(x)=\int_B\lim_{N\to\infty}f_N(x)dm(x)$$

where $x_0=x,\ f_N(x)=\sum_{n=0}^{N}\gamma^n G\circ C(x_n)$. Since $G\geq 0,\ f_N(x)\leq f_{N+1}(x)$ and using the monotone convergence theorem, we have

$$\int_B\lim_{N\to\infty}f_N(x)dm(x)=\lim_{N\to\infty}\int_{\mathcal{A}^c}f_N(x)dm_B(x)=\lim_{N\to\infty}\sum_{n=0}^{N}\langle\gamma^n G\circ C(x_n),m_B\rangle_{\mathcal{A}^c}$$

$$\lim_{N\to\infty}\sum_{n=0}^{N}\langle\gamma^n G\circ C(x_n),m_B\rangle_{\mathcal{A}^c}=\lim_{N\to\infty}\left\langle\mathbb{U}_C^1 G,\sum_{n=0}^{N}\gamma^n[\mathbb{P}_{T\circ C}^1 m_B]^n\right\rangle_{\mathcal{A}^c}$$

where we have used the fact that $x_n=(T\circ C)^n(x)$ and the duality between the Koopman operator $\mathbb{U}_{T\circ C}^1$ and the P-F operator $\mathbb{P}_{T\circ C}^1$. Let $\bar{\mu}_B^N=\sum_{n=0}^{N}\gamma^n[\mathbb{P}_{T\circ C}^1 m_B]^n$. The measure $\bar{\mu}_B^N$ is absolutely continuous with respect to Lebesgue measure m for all N. This follows because for any set $D\subset X_1$ if $m(D)=0$ then $([\mathbb{P}_{T\circ C}^1]^n m_B)(D)=m((T\circ C)^{-n}(D)\cap B)=0$ for all n and every set $B\subset X_1$. The latter is true because of the non-singularity assumption of the closed loop map $T\circ C$. Moreover $\bar{\mu}_B^N(D)\leq\bar{\mu}_B^{N+1}(D)$ for every set $D,B\subset X_1$. Hence there exists an integrable function $g_N(x)\geq 0$ such that $g_N(x)\leq g_{N+1}(x)$ and $d\bar{\mu}_B^N(x)=g_N(x)dm(x)$. So we have

$$\begin{aligned}\lim_{N\to\infty}\left\langle\mathbb{U}_C^1 G,\sum_{n=0}^{N}\gamma^n[\mathbb{P}_{T\circ C}^1 m_B]^n\right\rangle_{\mathcal{A}^c} &= \lim_{N\to\infty}\int_{\mathcal{A}^c}(\mathbb{U}_C^1 G)(x)g_N(x)dm(x)\\ &= \int_{\mathcal{A}^c}(\mathbb{U}_C^1 G)(x)\lim_{N\to\infty}g_N(x)dm(x)=\left\langle\mathbb{U}_C^1 G,\bar{\mu}_B\right\rangle_{\mathcal{A}^c}\end{aligned}$$

where $\bar{\mu}_B:=\sum_{n=0}^{\infty}\gamma^n[\mathbb{P}_{T\circ C}^1 m_B]^n=\sum_{n=0}^{\infty}\gamma^n[\mathbb{P}_T\cdot\mathbb{P}_C^1 m_B]^n]$ and $\bar{\mu}_B$ is known to be finite on any set $D\subset X_1$ because of a.e. stability property of the set $\mathcal{A}$ with geometric decay rate $\beta<\frac{1}{\gamma}$. Furthermore $\bar{\mu}_B$ satisfies following control Lyapunov measure equation

$$\gamma[\mathbb{P}_T^1\cdot\mathbb{P}_C^1\bar{\mu}_B](D)-\bar{\mu}_B(D)=-m_B(D) \tag{12}$$

for every set $D\in\mathcal{B}(X_1)$. Finally using the duality between $\mathbb{U}_C^1$ and $\mathbb{P}_C^1$, we get

$$\left\langle\mathbb{U}_C^1 G,\bar{\mu}_B\right\rangle_{\mathcal{A}^c}=\left\langle G,\mathbb{P}_C^1\bar{\mu}_B\right\rangle_{\mathcal{A}_c^u}$$

By appropriately selecting the measure on the right-hand side of the control Lyapunov measure equation (11) (i.e., m_B), stabilization of the attractor set with respect to a.e. initial conditions starting from a particular set can be studied. The minimum cost of stabilization is defined as the minimum over all a.e. stabilizing controller mappings, C, with a geometric decay as follows:

$$\mathcal{C}^*(B) = \min_{C} \mathcal{C}_C(B). \tag{13}$$

Next we write the infinite dimensional linear program for the optimal stabilization of the attractor set $\mathcal{A}$. Towards this goal, we first define the projection map, $P_1 : \mathcal{A}^c \times U \to \mathcal{A}^c$ as: $P_1(x, u) = x$, and denote the P-F operator corresponding to P_1 as $\mathbb{P}_{P_1} : \mathcal{M}(\mathcal{A}^c \times U) \to \mathcal{M}(\mathcal{A}^c)$, which can be written as $[\mathbb{P}^1_{P_1}\theta](D) = \int_{\mathcal{A}^c\times U} \chi_D(P_1(y))d\theta(y) = \int_{D\times U} d\theta(y) = \mu(D)$. Using this definition of projection mapping, P_1, and the corresponding P-F operator, we can write the linear program for the optimal stabilization of set B with unknown variable θ as follows:

$$\min_{\theta\geq 0} \langle G, \theta\rangle_{\mathcal{A}^c\times U}, \quad \text{s.t. } \gamma[\mathbb{P}^1_T\theta](D) - [\mathbb{P}^1_{P_1}\theta](D) = -m_B(D) \quad D \in \mathcal{B}(X_1). \tag{14}$$

Remark 11: Observe the geometric decay parameter satisfies $\gamma > 1$. This is in contrast to most optimization problems studied in the context of Markov-controlled processes, such as in Lasserre and Hernández-Lerma [5]. Average cost and discounted cost optimality problems are considered in [5]. The additional flexibility provided by $\gamma > 1$ guarantees the controller obtained from the finite dimensional approximation of the infinite dimensional program (14) also stabilizes the attractor set for system (7).

IV. Computational approach

The objective of the present section is to present a computational framework for the solution of the finite-dimensional approximation of the optimal stabilization problem in (14). There exists a number of references related to the solution of infinite dimensional linear programs (LPs), in general, and those arising from the control of Markov processes. Some will be described next. The monograph by Anderson and Nash [25] is an excellent reference on the properties of infinite dimensional LPs. Our intent is to use the finite-dimensional approximation as a tool to obtain stabilizing controls to the infinite-dimensional system. First, we will derive conditions under which solutions to the finite-dimensional approximation exist.

Following [18], [20], we discretize the state-space and control space for the purposes of computations as described below. Borrowing the notation from [20], let $\mathcal{X}_N := \{D_1, ..., D_i, ..., D_N\}$ denote a finite partition of the state-space $X \subset \mathbb{R}^q$. The measure space associated with $\mathcal{X}_N$ is $\mathbb{R}^N$. We assume without loss of generality that the attractor set, $\mathcal{A}$, is contained in D_N, that is, $\mathcal{A} \subseteq D_N$. Similarly, the control space, U, is quantized and the control input is assumed to take only finitely many control values from the quantized set, $\mathcal{U}_M = \{u^1, \ldots, u^a, \ldots, u^M\}$ where, $u^a \in \mathbb{R}^d$. The partition, $\mathcal{U}_M$, is identified with the vector space, $\mathbb{R}^{N\times M}$. The system map that results from choosing the controls u_N is denoted as T_{u_N} and the corresponding P-F operator is denoted as $P_{T_{u_N}} \in \mathbb{R}^{N\times N}$. Fixing the controls on all sets of the partition to u^a, i.e., $u_N(D_i) = u^a$, for all $D_i \in \mathcal{X}_N$, the system map that results is denoted as T_a with the corresponding P-F operator denoted as $P_{T_a} \in \mathbb{R}^{N\times N}$. The entries for P_{T_a} are calculated as: $(P_{T_a})_{(ij)} := \frac{m({T_a}^{-1}(D_j)\cap D_i)}{m(D_i)}$, where m is the Lebesgue measure and $(P_{T_a})_{(ij)}$ denotes the (i, j)-th entry of the matrix. Since $T_a : X \to X$, we have P_{T_a} is a Markov matrix. Additionally, $P^1_{T_a} : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ will denote the finite dimensional counterpart of the P-F operator restricted to $\mathcal{X}_N \setminus D_N$, the complement of the attractor set. It is easily seen that $P^1_{T_a}$ consists of the first $(N-1)$ rows and columns of P_{T_a}.

In [20], [18], stability analysis and stabilization of the attractor set are studied, using the above finite dimensional approximation of the P-F operator. The finite dimensional approximation of the P-F operator results in a weaker notion of stability, referred to as coarse stability [18].

With the above quantization of the control space and partition of the state space, the determination of the control $u(x) \in U$ (or equivalently $K(x)$) for all $x \in A^c$ has now been cast as a problem of choosing $u_N(D_i) \in \mathcal{U}_M$ for all sets $D_i \subset \mathcal{X}_N$. The finite dimensional approximation of the optimal stabilization problem (14) is equivalent to solving the following finite-dimensional LP:

$$\min_{\theta^a,\mu\geq 0} \sum_{a=1}^{M} (G^a)'\theta^a, \quad \text{s.t. } \gamma \sum_{a=1}^{M} (P_{T_a})'\theta^a - \sum_{a=1}^{M} \theta^a = -m, \tag{15}$$

where we have used the notation $(\cdot)'$ for the transpose operation, $m \in \mathbb{R}^{N-1}$ and $(m)_{(j)} > 0$ denote the support of Lebesgue measure, m, on the set D_j, $G^a \in \mathbb{R}^{N-1}$ is the cost defined on

$\mathcal{X}_N \setminus D_N$ with $(G^a)_{(j)}$ the cost associated with using control action u^a on set D_j; $\theta^a \in \mathbb{R}^{N-1}$ are, respectively, the discrete counter-parts of infinite-dimensional measure quantities in (14). In the LP (15), we have not enforced the constraint,

$$(\theta^a)_{(j)} > 0 \text{ for exactly one } a \in \{1, ..., M\}, \quad (16)$$

for each $j = 1, ..., (N-1)$. The above constraint ensures the control on each set in unique. We prove in the following the uniqueness can be ensured without enforcing the constraint provided the LP (15) has a solution. To this end, we introduce the dual LP associated with the LP in (15). The dual to the LP in (15) is,

$$\max_V \; m^{'}V, \quad \text{s.t. } V \leq \gamma P^1_{T_a} V + G^a \; \forall a = 1, ..., M. \quad (17)$$

In the above LP (17), V is the dual variables to the equality constraints in (15).

A. Existence of solutions to the finite LP

We make the following assumption throughout this section.

Assumption 12: There exists $\theta^a \in \mathbb{R}^{N-1} \; \forall \; a = 1, \ldots, M$, such that the LP in (15) is feasible for some $\gamma > 1$.

Note, Assumption 12 does not impose the requirement in (16). For the sake of simplicity and clarity of presentation, we will assume that the measure, m, in (15) is equivalent to the Lebesgue measure and $G > 0$. Satisfaction of Assumption 12 can be verified using the following algorithm.

Algorithm 1 1) Set $\mathcal{I} := 1, ..., N-1$, $\mathcal{I}_0 := N$, $L = 0$. 2) Set $\mathcal{I}_{L+1} := \emptyset$. 3) For each $i \in \mathcal{I} \setminus \{\mathcal{I}_0 \cup \ldots \cup \mathcal{I}_L\}$ do a) Pick the smallest $a \in 1, ..., M$ such that $(P_{T_a})_{(ij)} > 0$ for some $j \in \mathcal{I}_L$. b) If a exists then, set $u_N(D_i) := u^a$, $\mathcal{I}_{L+1} := \mathcal{I}_{L+1} \cup \{i\}$. 4) End For 5) If $\mathcal{I}_0 \cup \ldots \cup \mathcal{I}_L = \mathcal{I}$ then, set $L = L + 1$. STOP. 6) If $\mathcal{I}_{L+1} = \emptyset$ then, STOP. 7) Set $L = L + 1$. Go to Step 2.

The algorithm iteratively adds to $\mathcal{I}_{L+1}$, set D_i, which has a non-zero probability of transition to any of the sets in $\mathcal{I}_L$. In graph theory terms, the above algorithm iteratively builds a tree starting with the set $D_N \supseteq \mathcal{A}$. If the algorithm terminates in Step 6, then we have identified sets

$\mathcal{I} \setminus \{\mathcal{I}_0 \cup \ldots \cup \mathcal{I}_L\}$ that cannot be stabilized with the controls in $\mathcal{U}_M$. If the algorithm terminates at Step 5, then we show in the Lemma below that a set of stabilizing controls exist.

Lemma 13: Let $\mathcal{X}_N = \{D_1, \ldots, D_n\}$ be a partition of the state space, X, and $\mathcal{U}_M = \{u^1, \ldots, u^M\}$ be a quantization of the control space, U. Suppose the Algorithm 1 terminates in Step 5 after $L^{\max}$ iterations, then the controls u_N identified by the algorithm renders the system coarse stable.

Proof 14: Let $P_{T_{u_N}}$ represent the closed loop transition matrix resulting from the controls identified by Algorithm 1. Suppose $\mu \in \mathbb{R}^{N-1}$, $\mu \geq 0, \mu \neq 0$ be any initial distribution supported on the complement of the attractor set $\mathcal{X}_N \setminus D_N$. By construction μ has a non-zero probability of entering the attractor set after $L^{\max}$ transitions. Hence,

$$\sum_{i=1}^{N-1} (\mu'(P^1_{T_{u_N}})^{L^{\max}})_{(i)} < \sum_{i=1}^{N-1} (\mu)_{(i)} \implies \lim_{n\to\infty} (P^1_{T_{u_N}})^{nL^{\max}} \longrightarrow 0.$$

Hence, the sub-Markov matrix $P^1_{T_{u_N}}$ is transient and implies the claim.

Algorithm 1 is definitely less expensive than the approaches proposed in [20], where a mixed integer LP and a nonlinear programming approach were proposed. The strength of our algorithm is it is guaranteed to find deterministic stabilizing controls, if they exist. The following lemma shows an optimal solution to (15) exists under Assumption 12.

Lemma 15: Consider a partition $\mathcal{X}_N = \{D_1, \ldots, D_N\}$ of the state-space X with attractor set $\mathcal{A} \subseteq D_N$ and a quantization $\mathcal{U}_M = \{u^1, \ldots, u^M\}$ of the control space U. Suppose Assumption 12 holds for some $\gamma > 1$ and for $m, G > 0$. Then, there exists an optimal solution, θ, to the LP (15) and an optimal solution, V, to the dual LP (17) with equal objective values, $(\sum_{a=1}^{M} (G^a)'\theta^a = m'V)$ and θ, V bounded.

Proof 16: From Assumption 12, the LP (15) is feasible. Observe the dual LP in (17) is always feasible with a choice of $V = 0$. The feasibility of primal and dual linear programs implies the claim as a result of LP strong duality [26].

Remark 17: Note, existence of an optimal solution does not impose a positivity requirement on the cost function, G. In fact, even assigning $G = 0$ allows determination of a stabilizing control from the Lyapunov measure equation (15). In this case, any feasible solution to (15) suffices.

The next result shows the LP (15) always admits an optimal solution satisfying (16).

Lemma 18: Given a partition $\mathcal{X}_N = \{D_1, \ldots, D_N\}$ of the state-space, X, with attractor set, $\mathcal{A} \subseteq D_N$, and a quantization, $\mathcal{U}_M = \{u^1, \ldots, u^M\}$, of the control space, U. Suppose Assumption 12 holds for some $\gamma > 1$ and for $m, G > 0$. Then, there exists a solution $\theta \in \mathbb{R}^{N-1}$ solving (15) and $V \in \mathbb{R}^{N-1}$ solving (17) for any $\gamma \in [1, \overline{\gamma}_N)$. Further, the following hold at the solution: 1) For each $j = 1, ..., (N-1)$, there exists at least one $a_j \in 1, ..., M$, such that $(V)_{(j)} = \gamma(P^1_{T_{a_j}} V)_{(j)} + (G^{a_j})_{(j)}$ and $(\theta^{a_j})_{(j)} > 0$. 2) There exists a $\tilde{\theta}$ that solves (15), such that for each $j = 1, ..., (N-1)$, there is exactly one $a_j \in 1, ..., M$, such that $(\tilde{\theta}^{a_j})_{(j)} > 0$ and $(\tilde{\theta}^{a'})_{(j)} = 0$ for $a' \neq a_j$.

Proof 19:

From the assumptions, we have that Lemma 15 holds. Hence, there exists $\theta \in \mathbb{R}^{N-1}$ solving (15) and $V \in \mathbb{R}^{N-1}$ solving (17) for any $\gamma \in [1, \overline{\gamma}_N)$. Further θ and V satisfy following the first-order optimality conditions [26],

$$\begin{aligned} \sum_{a=1}^{M} \theta^a - \gamma \sum_{a=1}^{M} (P^1_{T_a})' \theta^a = m \\ V \leq \gamma P^1_{T_a} V + G^a \perp \theta^a \geq 0 \ \forall a = 1, ..., M. \end{aligned} \tag{18}$$

We will prove each of the claims in order.

Claim 1: Suppose, there exists $j \in 1, ..., (N-1)$ such that $(\theta^a)_{(j)} = 0$ for all $a = 1, ..., M$. Substituting in the optimality conditions (18) one obtains,

$$\gamma \sum_{a=1}^{M} ((P^1_{T_a})' \theta^a)_{(j)} = -(m)_{(j)}$$

which cannot hold since, $P^1_{T_a}$ has non-negative entries, $\gamma > 0$ and $\theta^a \geq 0$. Hence, there exists at least one a_j such that $(\theta^{a_j})_{(j)} > 0$. The complementarity condition in (18) then requires that $(V)_{(j)} = (\gamma P^1_{T_{a_j}} V)_{(j)} + (G^{a_j})_{(j)}$. This proves the first claim.

Claim 2: Denote $a(j) = \min\{a | (\theta^a)_{(j)} > 0\}$ for each $j = 1, ..., (N-1)$. The existence of $a(j)$ for each j follows from statement 1. Define $P^1_{T_u} \in \mathbb{R}^{(N-1)\times(N-1)}$ and $G^u \in \mathbb{R}^{N-1}$ as follows:

$$\begin{aligned} (P^1_{T_u})_{(ji)} &:= (P^1_{T_{a(j)}})_{(ji)} \ \forall i = 1, ..., (N-1) \\ (G^u)_{(j)} &:= (G^{a(j)})_{(j)} \end{aligned} \tag{19}$$

for all $j = 1, ..., (N-1)$. From the definition of $P^1_{T_u}$, G^u and the complementarity condition in (18) it is easily seen that V satisfies,

$$V = \gamma P^1_{T_u} V + G^u = \lim_{n\to\infty}((\gamma P^1_{T_u})^n V + \sum_{k=0}^{n}(\gamma P^1_{T_u})^k G^u). \tag{20}$$

Since V is bounded and $G^u > 0$ it follows that $\rho(P^1_{T_u}) < 1/\gamma$. Define $\tilde{\theta}$ as,

$$\begin{bmatrix} (\tilde{\theta}^{a(1)})_{(1)} \\ \vdots \\ (\tilde{\theta}^{a(N-1)})_{(N-1)} \end{bmatrix} = (I_{N-1} - \gamma(P^1_{T_u})')^{-1} m \tag{21a}$$

$$(\tilde{\theta}^a)_{(j)} = 0 \ \forall j = 1, ..., (N-1), \quad a \neq a(j). \tag{21b}$$

The above is well-defined since we have already shown that $\rho(P^1_{T_u}) < 1/\gamma$.

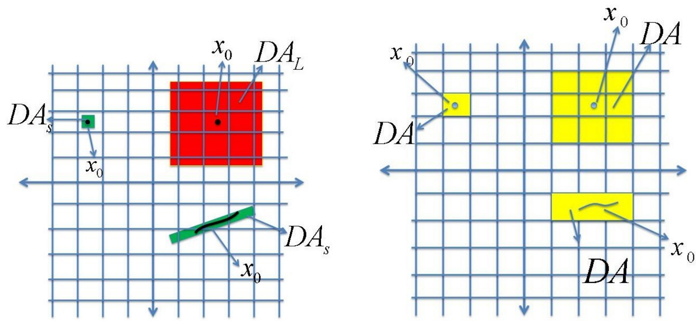
From the construction of $\tilde{\theta}$, we have that for each j there exists only one a_j, namely $a(j)$, for which $(\tilde{\theta}^{a(j)})_{(j)} > 0$. It remains to show that $\tilde{\theta}$ solves (15). For this observe that,

$$\begin{aligned} \sum_{a=1}^{M}(G^a)'\tilde{\theta}^a &\overset{(21b)}{=} \sum_{j=1}^{N-1}(G^{a(j)}_j \tilde{\theta}^{a(j)})_{(j)} \overset{(21a)}{=} (G^u)'(I_{N-1} - \gamma(P^1_{T_u})')^{-1} m \\ &= ((I_{N-1} - \gamma P^1_{T_u})^{-1} G^u)' m \overset{(20)}{=} V' m. \end{aligned} \tag{22}$$

The primal and dual objectives are equal with above definition of $\tilde{\theta}$ and hence, $\tilde{\theta}$ solves (15). The claim is proved.

The following theorem states the main result.

Theorem 20: Consider a partition $\mathcal{X}_N = \{D_1, \ldots, D_N\}$ of the state-space, X, with attractor set, $\mathcal{A} \subseteq D_N$, and a quantization, $\mathcal{U}_M = \{u^1, \ldots, u^M\}$, of the control space, U. Suppose Assumption 12 holds for some $\gamma > 1$ and for $m, G > 0$. Then, the following statements hold: 1) there exists a bounded θ, a solution to (15) and a bounded V, a solution to (17); 2) the optimal control for each set, $j = 1, ..., (N-1)$, is given by $u(D_j) = u^{a(j)}$, where $a(j) := \min\{a | (\theta^a)_{(j)} > 0\}$; 3) μ satisfying $\gamma(P^1_{T_u})'\mu - \mu = -m$, where $(P^1_{T_u})_{(ji)} = (P^1_{T_{a(j)}})_{(ji)}$ is the Lyapunov measure for the controlled system.



(a) (b)

Fig. 1: (a)Possible coarse stable set x_0 with small (green) and large (red) domain of attraction DA_s & DA_L respectively (b) Stable domains DA of x_0 not allowed by coarse stability of attractor set $\mathcal{A}$

Proof 21: Assumption 12 ensures that the linear programs (15) and (17) have a finite optimal solution (Lemma (15)). This proves the first claim of the theorem and also allows the applicability of Lemma 18. The remaining claims follow as a consequence.

Although the results in this section assumed the measure m is equivalent to Lebesgue, this can be easily relaxed to the case where m is absolutely continuous with respect to Lebesgue and is of interest where the system is not everywhere stabilizable. If it is known there are regions of the state-space not stabilizable, then m can be chosen such that its support is zero on these regions. If the regions are not known a priori then, (15) can be modified to minimize the l_1-norm of the constraint residuals. This is similar to the *feasibility phase* commonly employed in LP algorithms [27].

V. STABILITY IN FINITE DIMENSION

In this section we investigate the stability of the closed loop feedback controlled system upon application of the finite dimension control obtained as a solution of finite linear program (15) i.e., the stability of

$$x_{n+1} = T(x_n, K(x_n)) \qquad K(x) = u^{a(j)} \text{ for } x \in D_j \;\&\; u^{a(j)} \in \{u^1, ..., u^M\}. \tag{23}$$

It has been proved that ([18]; Theorem 26), while the existence of an infinite dimensional

Lyapunov measure implies a.e. stability of the attractor, the existence of a finite dimensional approximation of Lyapunov measure implies coarse stability of the attractor. Coarse stability of an attractor set is defined as follows.

Definition 22 (Coarse stability): Consider an attractor set $\mathcal{A} \subset X_0$ together with a finite partition $\mathcal{X}_1 = \{D_1, \ldots, D_L\}$ of the complement set $X_1 = X \setminus X_0$. The attractor set $\mathcal{A}$ is said to be coarsely stable w.r.t a.e initial conditions in X_1 if for an attractor set $B \subset W \subset X_1$, there exists no sub-partition $\mathcal{L} = \{D_{s_1}, D_{s_2}, \ldots, D_{s_l}\}$ in $\mathcal{X}_1$ with domain $S = \cup_{k=1}^{l} D_{s_k}$ such that $B \subset S \subset W$ and $T(S) \subset S$.

In Fig. 1a, we show the coarse stable behavior, with large (DA_L) and small (DA_s) [1] domain of attraction, that are possible in the complement of the attractor set i.e., the presence of these stable domain DA_L and DA_s in the complement set, $\mathcal{A}^c$, cannot be detected from the finite dimensional approximation of Lyapunov measure. On the other hand in Fig. 1b, we show stable domains DA that are not allowed by the coarse stability of attractor set. The difference between the stable regions depicted in the Fig. 1a and 1b is that while the stable regions in Fig. 1a are not contained in the entire cells of the finite partition $\mathcal{X}$, the stable regions in Fig. 1b is contained entirely in the cells of the finite partition.

Our objective is to design the controller so as to avoid the existence of coarse stable dynamics with large domain of attraction, i.e., DA_L. From our extensive numerical simulation we noticed that the presence of coarse stable dynamics with large domain of attraction, DA_L, are not completely unobservable and leave their signature on the finite dimensional closed loop Markov matrix. In particular, the lack of enough separation between the eigenvalue at one and the second largest eigenvalue of the closed loop Markov chain indicates the presence of stable dynamics with large domain of attraction in the complement set. To illustrate this point we consider the following

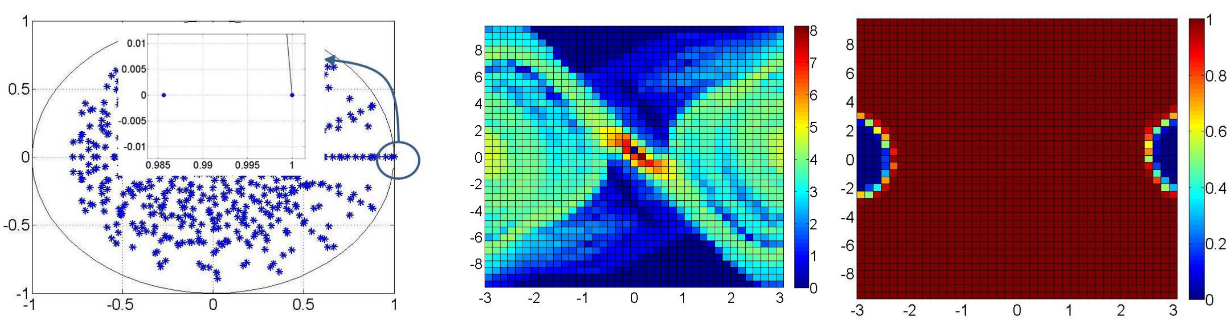
[1]Subscript L and s for DA are used for large and small domains for coarse stable dynamics in the complement set, $\mathcal{A}^c$, respectively.

inverted pendulum on cart example from [10]

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{\frac{g}{l}\sin x_1 - \frac{1}{2}m_r x_2^2 \sin 2x_1 - \frac{m_r}{m_l}\cos x_1 u}{\frac{4}{3} - m_r \cos^2 x_1} - 2\zeta\sqrt{\frac{g}{l}}x_2. \tag{24}$$

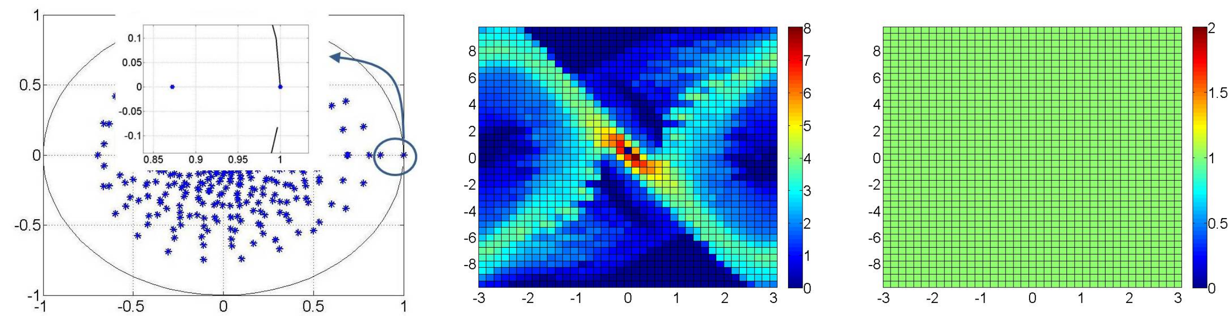
The parameter values used for simulation are $m_l = 2, m_r = \frac{m_l}{m_l+M}, M = 8, l = 0.5, g = 9.8$, and $\zeta = 0.2$. The objective is to stabilize the unstable saddle point at the origin while minimizing the cost function $G(x,u) = x_1^2 + x_2^2 + u^2$. For computational purpose, the discrete time dynamical system T is obtained from the continuous dynamics (24) by setting $T(x_1^0, x_2^0) = \phi(\delta t, x_1^0, x_2^0)$, where $\phi(t, x_1^0, x_2^0)$ is the solution of the system (24) starting from initial condition (x_1^0, x_2^0) with $\delta t = 0.1$. The state space X is chosen to be $[-\pi, \pi] \times [-10, 10]$ and is partitioned into 40×40 boxes. The control u takes finitely many values from the control set $\mathcal{U}_M = \{-80, -70, \ldots, -10, 0, 10, \ldots, 70, 80\}$.

The results in Figs. 2 and 3 are obtained by solving the linear program for parameter values of $\gamma = 1.00001$ and $\gamma = 1.05$ respectively. The parameter $\gamma > 1$ controls the rate of geometric decay of the initial measure (refer to Eqs. (14), (15), and Remark 11 for the role of parameter γ). We notice from Figs. 2a and 3a that the controllers designed with parameter value of $\gamma = 1.0001$ and $\gamma = 1.05$ lead to the spectral gap of 0.015 and 0.125 respectively between the first and the second largest eigenvalues of the closed loop Markov matrix.



(a) (b) (c)

Fig. 2: (a) Eigenvalues plot for closed loop Markov chain for $\gamma = 1.00001$; (b) Controlled Lyapunov measure; (c) Fraction of initial conditions from each box that asymptotically reach the origin.



(a) (b) (c)

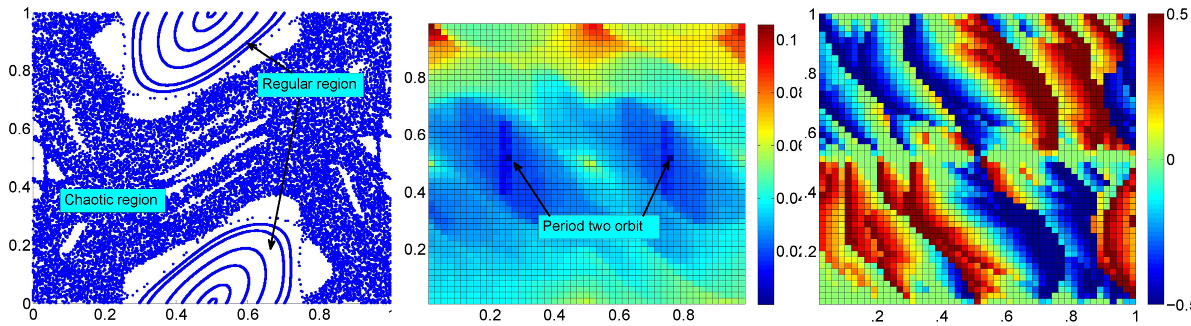
Fig. 3: (a) Eigenvalues plot for closed loop Markov chain for $\gamma = 1.05$; (b) Controlled Lyapunov measure; (c) Fraction of initial conditions from each box that asymptotically reach the origin.

For both these cases the closed loop Markov matrix is transient and is confirmed from the existence of the controlled Lyapunov measure as shown in Figs. 2b and 3b. In Figs. 2c and 3c, we show the plots for the fraction of initial conditions from each box of the partition that eventually end up at the origin. These plots are obtained by performing time domain simulation for closed loop system with finite dimensional control and with 900 initial conditions uniformly distributed in each box of the partition. In this sense, Figs. 2c and 3c reveal the true dynamics of the system after implementing the finite dimensional controller.

The regions in phase space near $(x, y) = (\pm 3, 0)$ in Fig. 2c indicates that the closed loop dynamics with finite dimensional control for parameter value of $\gamma = 1.00001$ is coarse stable with large domain of attraction. On the other hand, similar plot in Fig. 3c for $\gamma = 1.05$ shows that simulation starting from all the sample points from every box eventually end up at the origin and hence the closed loop dynamics is coarse stable with possible small domain of attraction. The important point to note is that the gap between the first and the second eigenvalue for the plot in Fig. 2a is almost zero and equals 0.015, whereas this gap is equal to 0.125 in Fig. 3c.

The existence of an eigenvalue close to one for the finite dimensional Markov chain corresponds to the presence of almost invariant region in the phase space [28]. Almost invariant regions are the regions in the phase space where system trajectories spend large amount of time before finally exiting these regions. For more details on almost invariant region and its connection to the second eigenvalue of the Markov matrix refer to [28], [29]. We argue that if there are stable regions with large domain of attraction in the complement of the stabilized invariant set and the closed loop sub-Markov matrix is transient then these stable regions appear as almost invariant region for the finite dimensional approximation. We conjecture that for fine enough partition the lack of enough separation between the first and the second eigenvalue of the closed loop Markov chain is a necessary condition for the presence of stable dynamics with large domain of attraction in the complement of the stabilized invariant set. In order to avoid the presence of such coarse stable dynamics with large domain of attraction the finite dimensional controller should be designed such that there is enough separation between the first and the second eigenvalues of the closed loop Markov matrix. The gap between the eigenvalues can be imposed by solving the finite dimensional linear program for larger values of γ if possible. The larger the value of γ, the more will be the separation between the first and the second eigenvalues of the closed loop Markov matrix.

On the other hand, stable regions with small domain of attraction, DA_s, as shown in Fig. 1a, are unobservable and leave no signature on the closed loop Markov matrix. However, the presence of such small stable region is not important from the point of view of any meaningful optimization as any small amount of noise or uncertainty will destroy such behavior. Rigorous connections between the finite dimensional approximation of the Markov chain and the true dynamics can be established using the stochastic stability results developed in [30], [31], [32], where the finite dimensional Markov matrix can be considered to be arising from the stochastic perturbation of the deterministic dynamical system. However, such rigorous theoretical connections are developed only for lower dimensional hyperbolic maps. Detailed investigations of such results for the controlled dynamical systems are beyond the scope of this paper and will be the topic of future investigation. In [20] (Section V), we have also discussed the effect of spatial discretization on the stability of closed loop systems via random perturbation.



(a) (b) (c)

Fig. 4: (a) Dynamics of the standard map; b) Optimal cost function; c) Optimal control input.

VI. Examples

The results in this section have been obtained using an interior-point algorithm, IPOPT [33]. IPOPT is an open-source software available through the COIN-OR repository [34], developed for solving large-scale non-convex nonlinear programs.

A. Standard Map

The standard map is a 2D map also known as the Chirikov map. It is one of the most widely studied map in dynamical systems [35]. The standard map is obtained by taking a Poincare map of one degree of freedom Hamiltonian system forced with a time-periodic perturbation and by writing the system in *action-angle* coordinates. The standard map forms a basic building block for studying higher degree of freedom Hamiltonian systems. The control standard map is described by following set of equations [35]:

$$x_{n+1} = x_n + y_n + Ku \sin 2\pi x_n \pmod 1, \quad y_{n+1} = y_n + Ku \sin 2\pi x_n \pmod 1, \tag{25}$$

where $(x, y) \in R := \{[0, 1) \times [0, 1)\}$ and u is the control input. The dynamics of the standard map with $u \equiv 1$ and $K = 0.25$ are shown in Fig. 4a. The phase space dynamics consists of a mix of chaotic and regular regions. With the increase in the perturbation parameter, K, the dynamics of the system becomes more chaotic. For the uncontrolled system (i.e., $u \equiv 0$), the entire phase space is foliated with periodic and quasi-periodic orbits. In particular, every $y =$

constant line is invariant under the system dynamics, and consists of periodic and quasi-periodic orbits for rational and irrational values of y, respectively. Our objective is to globally stabilize the period two orbit located at $(x, y) = (0.25, 0.5)$ and $(x, y) = (0.75, 0.5)$, while minimizing the cost function $G(x, y, u) = x^2 + y^2 + u^2$. To construct the finite dimensional approximation of the P-F operator, the phase space, R, is partitioned into 50×50 squares: each cell has 10 sample points. The discretization for the control space is chosen to be equal to $u^a = [-0.5 : 0.05 : 0.5]$. In Figs. 4b and 4c, we show the plot for optimal cost function and control input, respectively. We observe in Fig. 4c the control values used to control the system are approximately antisymmetric about the origin. This antisymmetry is inherent in the standard map and can also be observed in the uncontrolled standard map plot in Fig. 4c.

VII. Conclusions

Lyapunov measure is used for the optimal stabilization of an attractor set for a discrete time dynamical system. The optimal stabilization problem using a Lyapunov measure is posed as an infinite dimensional linear program. A computational framework based on set oriented numerical methods is proposed for the finite dimensional approximation of the linear program.

The set-theoretic notion of a.e. stability introduced by the Lyapunov measure offers several advantages for the problem of stabilization. First, the controller designed using the Lyapunov measure exploits the natural dynamics of the system by allowing the existence of unstable dynamics in the complement of the stabilized set. Second, the Lyapunov measure provides a systematic framework for the problem of stabilization and control of a system with complex non-equilibrium behavior.

VIII. Acknowledgment

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