Lyapunov measure for almost everywhere stability

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***Abstract*—This paper is concerned with analysis and computational methods for verifying global stability of an attractor set of a nonlinear dynamical system. Based upon a stochastic representation of deterministic dynamics, a Lyapunov measure is proposed for these purposes. This measure is shown to be a stochastic counterpart of stability (transience) just as invariant measure is a counterpart of attractor (recurrence). It is a dual of the Lyapunov function and is useful for the study of more general (weaker and set-wise) notions of stability. In addition to the theoretical framework, constructive methods for computing approximations to the Lyapunov measures are presented. These methods are based upon set-oriented numerical approaches. Several equivalent descriptions, including a series-formula and a system of linear inequalities, are provided for computational purposes. These descriptions allow one to carry over the intuition from the linear case with stable equilibrium to nonlinear systems with globally stable attractor sets. Finally, in certain cases exact relationship between Lyapunov functions and Lyapunov measures is also given.**

I. INTRODUCTION

For nonlinear dynamical systems, Lyapunov function based methods play a central role in both stability analysis and control synthesis [1]. Given the complexity of dynamic behavior possible even in low dimensions [2], these methods are powerful because they provide an analysis and design approach for *global* stability of an equilibrium solution. However, as opposed to linear systems, there are relatively few computational methods to construct these functions in general nonlinear settings and herein lies the barrier to their more widespread use. For nonlinear ODEs, two ideas have appeared in recent literature towards overcoming these barriers.

In [3], Rantzer introduced a dual to the Lyapunov function, referred to by the author as a **density function**, to define and study weaker notion of stability of an equilibrium solution of nonlinear ODEs. The author shows that the existence of a density function guarantees asymptotic stability in an almost everywhere sense, i.e., with respect to any *set* of initial conditions in the phase space with a positive Lebesgue measure. In the context of this paper, we note that Rantzer interprets his density function as "density of a substance that is **transported** along the system trajectories." The second idea involves computation of Lyapunov functions using sum of squares (SOS) polynomials; cf. [4] and [5] for an early work. This idea has recently appeared in the work of Parrilo [6], where the construction of Lyapunov function is cast as a *linear* semidefinite problem (or linear matrix inequalities, LMI) with suitable choice of polynomials (monomials) serving as a basis. LMI based methods and algorithms for verifying polynomial to be a SOS have appeared in [7], [8], [9], [10]; see also [4] and references therein. In a recent paper by [11], these two ideas have been combined to show that density formulation together with its computation using SOS methods leads to a convex and linear problem for the joint design of the density function and state feedback controller. In this paper, the three elements of **transport, linearity, and computations** are all shown to be intimately related to certain stochastic operators and their finite-dimensional discretizations.

The transport properties for ODEs, dynamical systems, or nonlinear continuous maps has a rich history of study using stochastic methods; cf., [12], [13]. Given a dynamical system, one can associate two different *linear* operators known as Koopman and Perron-Frobenius (P-F) operators. These two operators are adjoint to each other. While the dynamical system describes the evolution of an initial condition, the P-F operator describes the evolution of uncertainty in initial conditions. Under suitable technical conditions, the spectral analysis of the linear operators provides a description of the asymptotic dynamics of nonlinear dynamical systems. In particular, the eigenfunction with eigenvalue one characterizes the invariant sets capturing the long term asymptotic behavior of the system [14], [15]. Spectrum of these operator on the unit circle has the information about the cyclic behavior of the system [16], [17], [18]. More recently, there has been a significant interest in applied dynamical systems literature to develop finite-dimensional approximations of these operators for the computational analysis of global dynamics. Set-oriented numerical methods have been proposed for these purposes; cf. [19]. The stochastic operators together with their finite-dimensional approximations provide for the three elements of transport, linearity, and computations. In this paper, these and other properties of stochastic operators are used to develop extension of the ideas of [3] on one hand and propose a new set of linear computational tools for verifying stability on the other. In particular, there are three contributions of this paper that are discussed in the following three paragraphs.

First, it is shown that the duality expressed in the paper of [3] and linearity expressed in the paper of [11] is well-understood using stochastic methods. Spectral analysis of the stochastic operators is used to study the stability properties of the invariant sets of deterministic dynamical systems. In particular, we introduce **Lyapunov measure** as a dual to Lyapunov function. Lyapunov measure is closely related to Rantzer's density function, and like its counterpart it is shown to capture the weaker almost everywhere notion of stability.

This work was supported by research grants from the Iowa State University (UV) and the University of Illinois (PGM)

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Just as invariant measure is a stochastic counterpart of the invariant set, existence of Lyapunov measure is shown to give a stochastic conclusion on the stability of the invariant measure. The key advantage of relating Lyapunov measure to the P-F operator is a) the relationship serves to provide explicit formulas of the Lyapunov measure, and b) set-oriented methods can be used to compute it numerically.

For stable linear dynamical systems, the Lyapunov function can be obtained as a positive solution of the so-called Lyapunov equation. The equation is linear and the Lyapunov function is efficiently computed and can even be expressed analytically as an infinite-matrix-series expansion. For the series to converge, there exists a spectral condition on the linear dynamical system ($\rho(A) < 1$). The P-F formulation allows one to generalize these results to the study of stability of invariant and possibly chaotic attractor sets of nonlinear dynamical systems. More importantly, it provides a framework that allows one to carry over the intuition of the linear dynamical systems to nonlinear systems. For instance, the spectral condition is now expressed in terms of the P-F operator. The Lyapunov measure is shown to be a solution of a linear resolvent operator and admits an infinite-series expansion. The stability result, however, is typically weaker and one can only conclude stability in measure-theoretic (such as almost everywhere) sense. Finally, the non-negativity of the stochastic operator is shown to lead to a Linear Programming (LP) formulation for computing the Lyapunov measure.

The third contribution pertains to the formulation of these results in finite-dimensional settings. Using set-oriented numerical methods such as GAIO [19], the computation of approximate Lyapunov measure is cast as a solution to a finite system of linear inequalities. It is efficiently solved using Linear Programming. The finite-dimensional approximation is motivated by the computational concerns but as a by product leads to even weaker notions of stability. This notion is termed as **coarse stability** in this paper.

The outline of this paper is as follows. In Section II, preliminaries and notation from the dynamical systems literature related to P-F operator is reviewed. In Section III, Lyapunov measure is introduced and related to both the stochastic operators and certain notions of stability of an attractor set. In Sections IV and V, discrete approximation of the P-F operator and the Lyapunov measure respectively is given. The approximation is shown to be related to a certain weaker notion of stability, termed coarse stability, of the original dynamical system. In Section VI, relationship between the Lyapunov measure and function is provided. Finally, we conclude with a discussion on the merits of the approach in Section VII.

II. PRELIMINARIES AND NOTATION

In this paper, *discrete dynamical systems* or mappings of the form

$$x_{n+1} = T(x_n) \tag{1}$$

are considered. $T : X \to X$ is in general assumed to be only continuous and non-singular with $X \subset \mathbb{R}^n$, a compact set. A mapping T is said to be non-singular with respect to a measure m if $m(T^{-1}B) = 0$ for all $B \in \mathscr{B}(X)$ such that $m(B) = 0$. $\mathscr{B}(X)$ denotes the Borel σ-algebra on X, $\mathscr{M}(X)$ the vector space of real valued measures on $\mathscr{B}(X)$. Even though, deterministic dynamics are considered, stochastic approach is employed for their analysis. To aid this, some notations from the field of Ergodic theory is next introduced; cf., [13], [20].

A. Stochastic operators

In stochastic settings, the basic object of interest is a stochastic transition function:

Definition 1 (Stochastic transition function) *is a function* $p : X \times \mathscr{B}(X) \to [0,1]$ *such that*

1) $p(x,\cdot)$ *is a probability measure for every* $x \in X$,
2) $p(\cdot,A)$ *is Lebesgue-measurable for every* $A \in \mathscr{B}$.

Intuitively, $p(x,A)$ gives the probability for a transition from a point x into a set A. For Eq. (1), this probability is given by

$$p(x,A) = \delta_{T(x)}(A),$$

where δ is a Dirac measure. A stochastic transition function is used to define a linear operator on the space of measures $\mathscr{M}(X)$ as follows.

Definition 2 (Perron-Frobenius operator) *Let* $p(x,A)$ *be a stochastic transition function. The* Perron-Frobenius (P-F) operator $\mathbb{P} : \mathscr{M}(X) \to \mathscr{M}(X)$ *corresponding to* p *is defined by*

$$\mathbb{P}[\mu](A) = \int_X p(x,A)d\mu(x). \tag{2}$$

Borrowing terminology from applied probability theory [21], $\mathbb{P}$ will also be referred to as a *stochastic operator* with *transition kernel* $p(x,A)$. Since $p(x,X) = 1$, any stochastic operator necessarily satisfies

$$\mathbb{P}[\mu](X) = \int_X 1 d\mu(x) = \mu(X).$$

For the transition function $\delta_{T(x)}(\cdot)$ corresponding to a mapping T, the P-F operator is given by

$$\mathbb{P}[\mu](A) = \int_X \delta_{T(x)}(A)\, d\mu(x) = \int_X \chi_A(Tx)\, d\mu(x) = \mu(T^{-1}(A)),$$

where $\chi_A(\cdot)$ is the indicator function with support on A, and $T^{-1}(A)$ is the pre-image set:

$$T^{-1}(A) = \{x \in X : Tx \in A\}.$$

The more general form of the P-F operator in Eq. (2) is convenient for considering perturbations of the dynamical system in Eq. (1), useful for approximation and discretization purposes.

Definition 3 (Invariant measure) *is a measure* $\mu \in \mathscr{M}(X)$ *that satisfies*

$$\mu(A) = \int_X p(x,A)d\mu(x) \tag{3}$$

for all $A \in \mathscr{B}(X)$.

So, the invariant measures are the fixed-points of the P-F operator $\mathbb{P}$ that are additionally probability measures. From Ergodic theory, an invariant measure is always known to exist under the assumption that the mapping T is at least continuous and X is compact; [12].

Definition 4 (Koopman Operator) *For Eq. (1), the operator $U : C^0(X) \to C^0(X)$ defined by*

$$Uf(x) = f(Tx),$$

is called the Koopman operator with respect to the mapping T.

For $f \in C^0(X)$ and $\mu \in \mathscr{M}(X)$, define the inner product as

$$< f, \mu >:= \int_X f(x) d\mu(x).$$

With respect to this inner product, the Koopman operator is a dual to the P-F operator, where the duality is expressed by the following

$$< Uf, \mu >= \int_X Uf(x) d\mu(x) = \int_X f(x) d\mathbb{P}\mu(x) =< f, \mathbb{P}\mu > .$$

B. Attractor set and almost everywhere stability

In this paper, global stability properties of an attractor set will be investigated. Before stating the definition of attractor set, we state the following definition of $\omega-$limit set

Definition 5 (ω-limit set) *A point $y \in X$ is called a ω- limit point for a point $x \in X$ if there exists a sequence of integers $\{n_k\}$ such that $T^{n_k}(x) \to y$ as $k \to \infty$. The set of all ω-limit point for x is denoted by $\omega(x)$ and is called its ω- limit set.*

A set $A \subset X$ is called *T-invariant* if

$$T(A) = A. \tag{4}$$

Definition 6 (Attractor set) *A close T-invariant set $A \subset X$ is said to be an attractor set if it satisfies the following two properties*

1) *there exists a neighborhood $V \subset X$ of A such that $\omega(x) \subset A$ for almost everywhere (a.e.) $x \in V$ with respect to finite measure $m \in \mathscr{M}(X)$. V is called the local neighborhood of A.*
2) *there is no strictly smaller closed set $A' \subset A$ which satisfies property 1.*

The notation $A \subset V \subset X$ is used to denote an attractor set A with local neighborhood V in X.

Remark 7 Measure m can typically be taken to be a Lebesgue measure.

There are various definitions of attractor set in the dynamical systems literature; Ch. 1 of [22] or the introduction in [23]. The above definition of attractor set is due to Milnor and appears in [23]. The important point of the definition is that it does not require the local stability (in the sense of Lyapunov) of the invariant set A and hence allows for a broad class of attractor sets. Using Eq. (3), a measure $\mu \neq 0$ is said to be a T-invariant measure if

$$\mu(B) = \mu(T^{-1}(B)) \tag{5}$$

for all $B \in \mathscr{B}(A)$. A T-invariant measure in Eq. (5) is a stochastic counterpart of the T-invariant set in Eq. (4) [2], [12]. For typical dynamical systems, the set A equals the support of *its* invariant measure μ. Now we state some measure-theoretic preliminaries and definition of almost everywhere stability of an attractor set.

Definition 8 (Absolutely continuous measure) *A measure μ is absolutely continuous with respect to another measure ϑ, denoted as $\mu \prec \vartheta$, if $\mu(B) = 0$ for all $B \in \mathscr{B}(X)$ with $\vartheta(B) = 0$.*

Definition 9 (Equivalent measure) *Two measures μ and ϑ are equivalent ($\mu \approx \vartheta$) provided $\mu(B) = 0$ if and only if $\vartheta(B) = 0$ for $B \in \mathscr{B}(X)$.*

Definition 10 (Almost everywhere stable) *An attractor set A for the dynamical system $T : X \to X$ is said to be stable almost everywhere (a.e.) with respect to a finite measure $m \in \mathscr{M}(A^c)$ if*

$$m\{x \in A^c : \omega(x) \not\subseteq A\} = 0$$

For the special case of a.e. stability of an equilibrium point x_0 with respect to the Lebesgue measure, the definition reduces to

$$Leb\{x \in X : \lim_{n\to\infty} T^n(x) \neq x_0\} = 0,$$

where Leb in this case is the Lebesgue measure. Motivated by the familiar notion of point-wise exponential stability in phase space, we introduce a stronger notion of stability in the measure space. This stronger notion of stability captures a geometric decay rate of convergence.

Definition 11 (Almost everywhere stable with geometric decay) *The attractor set $A \subset X$ for the dynamical system $T : X \to X$ is said to be stable almost everywhere with geometric decay with respect to a finite measure $m \in \mathscr{M}(A^c)$ if given $\varepsilon > 0$, there exists $K(\varepsilon) < \infty$ and $\beta < 1$ such that*

$$m\{x \in A^c : T^n(x) \in B\} < K\beta^n \quad \forall\, n \geq 0$$

for all sets $B \in \mathscr{B}(X \setminus U(\varepsilon))$, where $U(\varepsilon)$ is the ε neighborhood of the attractor set A.

Remark 12 In the definition of almost everywhere stability and almost everywhere stability with geometric decay with respect to measure m, it is implied that condition 1. in the definition of attractor set (Def. 6) holds true with respect to the measure m as well.

C. Stochastic Analysis

Study of Markov chains on finite or countable sets is by now a well-established discipline in applied probability theory; cf. [24], [25], [26]. Results on the stability or Ergodic properties of these Markov chains in more general settings appear in recent monographs of [21], [27]. Many a results appearing in this paper are motivated by this literature. The stochastic transition function $p(x,A)$ is referred to as a *transition kernel* [28], [27] or a Markov transition function [21]. Borrowing notation from [21], the two linear operators of interest are recognized as

$$\begin{aligned} \mathbb{P}[\mu](A) &= \int_X \mu(dx)p(x,A), \\ Uf(x) &= \int_X p(x,dy)f(y). \end{aligned}$$

III. STABILITY IN INFINITE-DIMENSION

In this section, Lyapunov type global stability conditions are presented using the infinite-dimensional P-F operator $\mathbb{P}$ for the mapping $T: X \to X$ in Eq. (1). Recall that an attractor set A is defined to be globally stable with respect to a measure m if

$$\omega(x) \subseteq A, \text{ a.e. } x \in A^c,$$

where $a.e.$ is with respect to the measure m. Now, consider the restriction of the mapping $T: A^c \to X$ on the complement set. This restriction can be associated with a suitable stochastic operator related to $\mathbb{P}$ that is useful for the stability analysis with respect to the complement set. The following section makes the association precise.

A. Decomposition of Perron-Frobenius operator

Definition 13 (Sub-stochastic transition function) *is a function $p: X \times \mathscr{B}(X) \to [0,1]$ such that*

1) *$p(x,\cdot)$ is a real-valued measure with $p(x,X) \le 1$ for $x \in X$,*
2) *$p(\cdot,A)$ is Lebesgue-measurable for every $A \in \mathscr{B}$.*

The associated linear operator, with transition kernel $p(x,A)$, is called a *sub-stochastic operator* [21]. In this section, it will be shown that

1) the dynamical system corresponding to the mapping $T: A^c \to X$ defines a sub-stochastic operator $\mathbb{P}_1$ acting on $\mathscr{M}(A^c)$, and
2) $\mathbb{P}$ on $\mathscr{M}(X)$ and $\mathbb{P}_1$ are related.

Consider $T: A^c \cup A \to X$ such that A is left invariant by T, i.e., $T: A \to A$. For $B \subset \mathscr{B}(A^c)$,

$$\mathbb{P}[\mu](B) = \int_X \delta_{T(x)}(B)d\mu(x) = \int_{A^c} \delta_{T(x)}(B)d\mu(x),$$

because $T(x) \in B$ implies $x \notin A$. Thus, corresponding to the mapping $T: A^c \to X$, the operator

$$\mathbb{P}_1[\mu](B) := \int_{A^c} \delta_{T(x)}(B)d\mu(x) = \mu(T^{-1}B \cap A^c) \qquad (6)$$

is well-defined for $\mu \in \mathscr{M}(A^c)$ and $B \subset \mathscr{B}(A^c)$. Next, the restriction $T: A \to A$ can also be used to define a P-F operator denoted by

$$\mathbb{P}_0[\mu](B) = \int_B \delta_{T(x)}(B)d\mu(x),$$

where $\mu \in \mathscr{M}(A)$ and $B \subset \mathscr{B}(A)$.

The above considerations suggest a representation of the P-F operator $\mathbb{P}$ in terms of $\mathbb{P}_0$ and $\mathbb{P}_1$. This is indeed the case if one considers a splitting of the measure space

$$\mathscr{M}(X) = \mathscr{M}_0 \oplus \mathscr{M}_1, \qquad (7)$$

where $\mathscr{M}_0 := \mathscr{M}(A)$ and $\mathscr{M}_1 := \mathscr{M}(A^c)$. Note that $\mathbb{P}_0 : \mathscr{M}_0 \to \mathscr{M}_0$ because $T: A \to A$ and $\mathbb{P}_1 : \mathscr{M}_1 \to \mathscr{M}_1$ by construction. Let $\Pi : \mathscr{M} \to \mathscr{M}_0$ denote the projection operator *onto* $\mathscr{M}_0$,

$$\Pi\mu(B) = \mu(B \cap A) \quad \& \quad (I-\Pi)\mu(B) = \mu(B \cap A^c).$$

The following is then easily seen:

$$\Pi\mathbb{P}\Pi = \mathbb{P}_0, \quad (I-\Pi)\mathbb{P}(I-\Pi) = \mathbb{P}_1, \quad \& \quad (I-\Pi)\mathbb{P}\Pi = 0.$$

It then follows that on the splitting defined by Eq. (7), the P-F operator has a lower-triangular matrix representation given by

$$\mathbb{P} = \begin{bmatrix} \mathbb{P}_0 & 0 \\ \times & \mathbb{P}_1 \end{bmatrix}. \qquad (8)$$

The invariant measure defined with respect to the operator $\mathbb{P}_0$ is a stochastic counterpart of the attractor set supported on the set A. Analogously, the stability conditions are expressed in terms of a certain sub-invariant measure that is defined with respect to the sub-stochastic operator $\mathbb{P}_1$. This is the subject of the following section.

B. Stability & Lyapunov measure

The lower triangular representation of $\mathbb{P}$ in Eq. (8) is convenient because then

$$\mathbb{P}^n = \begin{bmatrix} \mathbb{P}_0^n & 0 \\ \times & \mathbb{P}_1^n \end{bmatrix}, \qquad (9)$$

where $\mathbb{P}_1^n = (I-\Pi)\mathbb{P}^n(I-\Pi)$. More explicitly, for $B \subset \mathscr{B}(A^c)$,

$$\begin{aligned} \mathbb{P}_1\mu(B) &= \int_{A^c} \chi_B(Tx)d\mu(x) = \mu(T^{-1}(B) \cap A^c) & (10) \\ \mathbb{P}_1^n\mu(B) &= \int_{A^c} \chi_B(T^n x)d\mu(x) = \mu(T^{-n}(B) \cap A^c). & (11) \end{aligned}$$

These formulas are useful because one can now express the conditions for stability in Definitions 10 and 11 in terms of the asymptotic behavior of the operator $\mathbb{P}_1^n$.

Lemma 14 *Let $T : X \to X$ in Eq. (1) be a non-singular mapping with respect to measure m with an attractor set $A \subset V \subset X$ with its local neighborhood V, $U(\varepsilon)$ is an ε-neighborhood of A, and $A^c = X \setminus A$. The following express conditions for a.e. stability with respect to a finite measure $m \in \mathscr{M}(A^c)$:*

1) *The attractor set A is a.e. stable (definition 10) with respect to a measure m if and only if*

$$\lim_{n\to\infty} \mathbb{P}_1^n m(B) = 0 \qquad (12)$$

for all sets $B \in \mathscr{B}(X \setminus U(\varepsilon))$ and every $\varepsilon > 0$.

2) *The attractor set A is a.e. stable with geometric decay (definition 11) with respect to a measure m if and only*

if for every $\varepsilon > 0$, there exists $K(\varepsilon) < \infty$ and $\beta < 1$ such that

$$\mathbb{P}_1^n m(B) < K\beta^n \quad \forall\, n \geq 0 \tag{13}$$

and for all sets $B \in \mathscr{B}(X \setminus U(\varepsilon))$.

Proof: For $B \in \mathscr{B}(X \setminus U(\varepsilon))$, denote

$$B_n = \{x \in A^c \text{ and } T^n(x) \in B\}.$$

It is then easy to see that

$$m(B_n) = m(T^{-n}(B) \cap A^c) = \mathbb{P}_1^n m(B),$$

where the last equality follows from Eq. (11). The equivalence for part 2 (Eq. (13)) then follows by applying definition 11. To see part 1, note that

$$\lim_{n\to\infty} \chi_{B_n}(x) = 0 \tag{14}$$

for all x whose ω-limit points lie in A. If A is assumed a.e. stable, the limit in Eq. (14) is a.e. zero and

$$0 = \int_{A^c} \lim_{n\to\infty} \chi_{B_n}(x)dm(x) = \lim_{n\to\infty} \int_{A^c} \chi_{B_n}(x)dm(x) = \lim_{n\to\infty} \mathbb{P}_1^n m(B)$$

by dominated convergence theorem; cf., [29]. Conversely, let A be an attractor with some local neighborhood V. For $\varepsilon > 0$, consider the set

$$S_n = \{x \in A^c : T^k(x) \in X \setminus U(\varepsilon) \ \text{ for some } \ k > n\}$$

and let

$$S = \cap_{n=1}^{\infty} S_n$$

i.e., S is the set of points, some of whose limit points lie in $X \setminus U(\varepsilon)$. For a.e. stability, we need to prove that $m(S) = 0$. Let $\tilde{S} := S \cap (X \setminus V)$, then by the property of the local neighborhood $m(S) = m(\tilde{S})$. So, we prove the result by showing that $m(\tilde{S}) = 0$. Clearly, $x \in S_n$ if and only if $T(x) \in S_{n-1}$. By construction, $x \in S$ if and only if $T(x) \in S$, i.e., $S = T^{-1}(S)$. Furthermore, $S \subset A^c$ and we have,

$$m(S) = m(T^{-1}(S) \cap A^c) = \mathbb{P}_1 m(S). \tag{15}$$

Now, $\tilde{S} \subset S$ with $m(\tilde{S}) = m(S)$. Since T is non-singular, this implies that $\mathbb{P}_1 m(\tilde{S}) = \mathbb{P}_1 m(S)$ and using Eq. (15),

$$m(\tilde{S}) = \mathbb{P}_1 m(\tilde{S}), \tag{16}$$

where $\tilde{S} \subset X$ lies outside some local neighborhood of A. If $\lim_{n\to\infty} \mathbb{P}_1^n m(B) = 0$ for all $B \in \mathscr{B}(X \setminus U(\varepsilon))$ and in particular for $B = \tilde{S}$ then Eq. (16) implies that $m(\tilde{S}) = 0$ and thus $m(S) = 0$. Since ε here is arbitrary, we have

$$m\{x \in A^c : \omega(x) \not\subseteq A\} = 0,$$

and thus A is a.e. stable in the sense of definition 10. ■

The two conditions in Eq. (12)-(13) represent a certain property, **transience**, of the stochastic operator $\mathbb{P}_1$ with respect to Lebesgue measure m. For stability verification, the two conditions in by themselves are not any more useful than the definitions themselves. The definition involves iterating the mapping for *all* initial conditions in A^c while the two conditions involve iterating the stochastic operator for *all* Borel set B in A^c. Both are equally complex. However, just as stability can be verified by constructing Lyapunov function for the mapping T, transience can be verified by constructing a **Lyapunov measure** for the operator $\mathbb{P}$.

Definition 15 (Lyapunov measure) *is any non-negative measure $\bar{\mu} \in \mathscr{M}(A^c)$ which is finite on $\mathscr{B}(X \setminus U(\varepsilon))$ and satisfies*

$$\mathbb{P}_1 \bar{\mu}(B) < \alpha \bar{\mu}(B), \tag{17}$$

for every set $B \subset \mathscr{B}(X \setminus U(\varepsilon))$ and for every $\varepsilon > 0$ where

$$\bar{\mu}(B) > 0.$$

$\alpha \leq 1$ is some positive constant.

This construction and the Lyapunov measure's relationship with the two notions of transience will be a subject of the following three theorems. The first theorem shows that the existence of a Lyapunov measure $\bar{\mu}$ is sufficient for almost everywhere stability with respect to any absolutely continuous measure m.

Theorem 16 *Consider $T : X \to X$ in Eq. (1) with an attractor set $A \subset V \subset X$. Suppose there exists a Lyapunov measure $\bar{\mu}$ (Definition (15)) with $\alpha = 1$, then the attractor set A is almost everywhere stable with respect to measure any finite measure m that is equivalent to Lyapunov measure $\bar{\mu}$.*

Proof: Consider any set $B \in \mathscr{B}(X \setminus U(\varepsilon))$ with $m(B) > 0$. Using Lemma 14, a.e. stability is equivalent to

$$\lim_{n\to\infty} \mathbb{P}_1^n m(B) = 0. \tag{18}$$

To show Eq. (18), it is first claimed that $\lim_{n\to\infty} \mathbb{P}_1^n \bar{\mu}(B) = 0$. Since $m \prec \bar{\mu}$, the claim implies Eq. (18) and thus a.e. stability. To prove the claim, we note that $\bar{\mu}(B) > 0$ and consider the sequence of real numbers $\{\mathbb{P}_1^n \bar{\mu}(B)\}$. Using the definition of Lyapunov measure (eqn. 17), this is a decreasing sequence of non-negative numbers. Its limit is shown to be zero by repeating the argument in Lemma 14. In particular, let

$$S := \{x \in A^c : \lim_{n_k\to\infty} T^{n_k}(x) \in B\}$$

be the set of points, some of whose ω-limit points lie in B. For $B_n = \{x \in A^c : T^n x \in B\}$, $\chi_{B_n}(x) \to 0$ whenever $x \notin S$. By dominated convergence theorem,

$$\lim_{n\to\infty} \mathbb{P}_1^n \bar{\mu}(B) = \lim_{n\to\infty} \int_{A^c} \chi_{B_n}(x) d\bar{\mu}(x) \leq \bar{\mu}(S). \tag{19}$$

As in Lemma 14, it follows that $T^{-1}(S) = S$, $\mathbb{P}_1 \bar{\mu}(S) = \bar{\mu}(S)$, which together with the property of the local neighborhood V and Lyapunov measure gives $\bar{\mu}(S) = 0$. Using Eq. (19),

$$\lim_{n\to\infty} \mathbb{P}_1^n \bar{\mu}(B) = 0,$$

and this verifies the claim and thus proves the theorem. ■

The following theorem provides a sufficient condition for almost everywhere stability with geometric decay in terms of Lyapunov measure.

Theorem 17 *Consider $T : X \to X$ in Eq. (1) with an attractor set $A \subset V \subset X$. Suppose there exists a Lyapunov measure (Definition 15) with $\alpha < 1$, then*

1) *A is a.e. stable with respect to any finite measure m which is absolutely continuous with respect to Lyapunov measure $\bar{\mu}$.*
2) *A is a.e. stable with geometric decay with respect to any measure m satisfying $m \leq \gamma\bar{\mu}$ for some constant $\gamma > 0$.*

Proof:

1) Using definition (15) of the Lyapunov measure with $\alpha < 1$, we get

$$\mathbb{P}_1^n\bar{\mu}(B) < \alpha^n\bar{\mu}(B) \quad \text{which implies} \quad \lim_{n\to\infty}\mathbb{P}_1^n\bar{\mu}(B) = 0$$

Since $m \prec \bar{\mu}$, we have

$$\lim_{n\to\infty}\mathbb{P}_1^n m(B) = 0$$

the proof then follows from Lemma (14)

2) Consider any set $B \in \mathscr{B}(X \setminus U(\varepsilon))$. A simple calculation shows that

$$\mathbb{P}_1^n m(B) \leq \gamma\mathbb{P}_1^n\bar{\mu}(B) \leq \alpha^n\gamma\bar{\mu}(B) < K\alpha^n,$$

where $K(\varepsilon) = \gamma\bar{\mu}(X \setminus U(\varepsilon))$ is finite. Using Lemma 14, A is stable almost everywhere with geometric decay with respect to the measure m.

■

With the stronger stability condition of geometric decay, one can construct Lyapunov measure $\bar{\mu}$ as an infinite series involving sub-Markov operator $\mathbb{P}_1$. This leads to the following theorem:

Theorem 18 *Let $T : X \to X$ in Eq. (1) be a non-singular mapping with respect to finite measure m, with an attractor set $A \subset V \subset X$, $U(\varepsilon)$ is an ε-neighborhood of A, and $A^c = X \setminus A$. Suppose A is stable a.e. with geometric decay with respect to measure $m \in \mathscr{M}(A^c)$. Then there exists a Lyapunov measure $\bar{\mu}$ with $\alpha = 1$ such that Lyapunov measure is equivalent to measure m ($\bar{\mu} \approx m$). Furthermore, $\bar{\mu}$ may be constructed to dominate measure m i.e., $m(B) \leq \bar{\mu}(B)$.*

Proof: For any given $\varepsilon > 0$, construct a measure $\bar{\mu}$ as:

$$\bar{\mu}(B) = (I + \mathbb{P}_1 + \mathbb{P}_1^2 + \ldots)m(B) = \sum_{j=0}^{\infty}\mathbb{P}_1^j m(B), \tag{20}$$

where $B \in \mathscr{B}(X \setminus U(\varepsilon))$. For such sets, the geometric decay stability condition (see definition 11) implies that there exists a $K(\varepsilon) < \infty$ and $\beta < 1$ such that

$$\mathbb{P}_1^n m(B) < K\beta^n.$$

As a result, the infinite-series in Eq. (20) converges, and $\bar{\mu}(B)$ is well-defined, non-negative, and finite. Since, T is assumed non-singular with respect to measure m, the individual measures $\mathbb{P}_1^n m$ are absolutely continuous with respect to m and thus $\bar{\mu} \prec m$. From the construction of the Lyapunov measure it follows that

$$m(B) \leq \bar{\mu}(B),$$

and thus the two measures are equivalent Applying $(\mathbb{P}_1 - I)$ to both sides of Eq. (20), we get

$$\mathbb{P}_1\bar{\mu}(B) - \bar{\mu}(B) = -m(B) < 0 \text{ implies } \mathbb{P}_1\bar{\mu}(B) < \bar{\mu}(B)$$

whenever $m(B) > 0$, and equivalently, $\bar{\mu}(B) > 0$. ■

Remark 19 In the three theorems presented above, A is a.e. stable with respect to $m \in \mathscr{M}(A^c)$. In general, m can be any finite measure. Our primary interest is in Lebesgue a.e. stability, and we often take m to be the Lebesgue measure. Another finite measure of interest is

$$m_S(B) = m(B \cap S) \tag{21}$$

where $A \subset S \subset X$, $B \in \mathscr{B}(X \setminus U(\varepsilon))$, and m is the Lebesgue measure. Note that measure m_S in this case is not necessarily a non-singular measure with respect to T, however measure m_S can be used to 1) study local stability with respect to the initial conditions in $S \subset X$ and 2) characterize the domain of attraction of any invariant set A.

Before closing this section, we summarize the salient features of the Lyapunov measure:

1) its existence allows one to verify a.e. asymptotic stability (Theorem 17),
2) for an asymptotically stable system with geometric decay, the infinite-series (see Eq. (20))

$$(I - \mathbb{P}_1)^{-1}m = (I + \mathbb{P}_1 + \ldots + \mathbb{P}_1^N + \ldots)m \tag{22}$$

can be used to construct it.

The series-formulation in fact is related to the well-known Lyapunov equation in linear settings.

C. Lyapunov function and Koopman operator

Consider a linear dynamical system

$$x(n+1) = Ax(n),$$

where $\rho(A) < 1$. With a Lyapunov function candidate $V(x) = x'Px$, the Lyapunov equation is $A'PA - P = -Q$,

where Q is positive definite. A positive-definite solution for P is given by

$$P = Q + A'QA + \ldots + A'^nQA^n + \ldots,$$

where the series converges iff $\rho(A) < 1$. Setting $g(x) = x'Qx$, the infinite-series solution for any $x \in \mathbb{R}^n$ is given by

$$V(x) = \sum_{n=0}^{\infty} g(A^n x) = \sum_{n=0}^{\infty}(U^n g)(x) \tag{23}$$

where U is the Koopman operator, the dual to $\mathbb{P}$. The choice of $g(0) = 0$ on the complement set to the attractor $\{0\}$ ensures that the series representation converges. Even though, we have arrived at the series representation in Eq. (23) starting from the linear settings, the series is valid for nonlinear dynamical system or continuous mapping of Eq. (1); U is the Koopman operator for mapping T. If the series converges, one can express the solution in terms of the resolvent operator as in Eq. (23). For a convergent series, it is also easy to check that

$$V(Tx) - V(x) = UV(x) - V(x) = -g(x), \tag{24}$$

i.e., V is a Lyapunov function for $g(x) > 0$. Note that the function g need not be quadratic or even a polynomial – any positive C^0 function with $g(0) = 0$ will suffice. Moreover, the description is linear. The following theorem shows that the Lyapunov function can be constructed by using the resolvent of the Koopman operator for a stable system. In particular, we assume that the equilibrium point is globally exponentially stable and prove in essence a converse Lyapunov theorem for stable systems; cf., [1].

Theorem 20 *Consider $T : X \to X$ as in Eq. (1). Suppose $x = 0$ is a fixed-point ($T(0) = 0$), which is globally exponentially stable, i.e.,*

$$\parallel T^n(x) \parallel \leq K\alpha^n \parallel x \parallel \quad \forall x \in X \tag{25}$$

where $\alpha < 1$, $K > 1$, and $\|\cdot\|$ is the Euclidean norm in X. Then there exists a non-negative function $V : X \to \mathbb{R}^+$ satisfying

$$\begin{aligned} a \parallel x \parallel^p &\leq V(x) \leq b \parallel x \parallel^p, \\ V(Tx) &\leq c \cdot V(x), \end{aligned}$$

where a, b, c, p are positive constants; $c < 1$. Also, V can be expressed as

$$V(x) = (I - U)^{-1} f(x),$$

where $f(x) = \parallel x \parallel^p$ and U is the koopman operator corresponding to the dynamical system T.

Proof: Let $f(x) = \parallel x \parallel^p$ with $p \geq 1$ and set

$$V_N(x) = \sum_{n=0}^{N} f(T^n x) = \sum_{n=0}^{N} U^n f(x). \tag{26}$$

Now,

$$\|V_N(x)\| \leq \sum_{n=0}^{\infty} \parallel T^n x \parallel^p \leq K^p \sum_{n=0}^{\infty} \alpha^{np} \parallel x \parallel^p \leq \frac{K^p}{1 - \alpha^p} \|x\|^p \tag{27}$$

satisfies a uniform bound because of globally exponentially stability (Eq. (25)) and because X is compact. As a result, $V(x) = \lim_{N\to\infty} V_N(x)$ converges point-wise and the limit is well-defined and can be expressed as an infinite-series,

$$V(x) = \lim_{N\to\infty} \sum_{n=0}^{N} U^n f(x) = (I - U)^{-1} f(x). \tag{28}$$

By Eqs. (26) and (27),

$$\parallel x \parallel^p \leq V(x) \leq \frac{K^p}{1 - \alpha^p} \parallel x \parallel^p = b \parallel x \parallel^p,$$

where $b > 1$. Finally, because $T : X \to X$, $V(Tx) = UV(x)$, Eq. (28) gives

$$(U - I)V(x) = -f(x) = - \parallel x \parallel^p \leq \frac{-1}{b} \cdot V(x) \tag{29}$$

Set $c = (1 - \frac{1}{b})$. Clearly, $c < 1$ and using Eq. (29),

$$V(Tx) \leq c \cdot V(x).$$

■

The series formulation in Eq. (22) using the P-F operator on the complement set A^c is a dual to the series expansion using the Koopman operator in Eq. (28). The Lyapunov *measure* description thus is a dual to the Lyapunov *function* description. The measure-theoretic description provides a *set-wise* counterpart to the *point-wise* description with Lyapunov function. One of the advantage is that weaker notions of stability, such as a.e stability are possible with measure-theoretic description. The other advantage is that Lyapunov measures may be computed for stability verification and control design using much the same set-oriented methods as are used for computation of invariant measures. This will be a subject of the following two sections. We note that an invariant measure, a stochastic object, is perhaps the simplest notion to capture *recurrence* of an attractor set. The point-wise or the topological description of the same is complex. Likewise, we conjecture that Lyapunov measure is the natural stochastic counterpart of *transience* of the complement set (stability of an attractor set). As the following sections show, the approximation of the Lyapunov measure for nonlinear systems is possible using *linear algorithms*. These can be viewed as generalizations to constructing Lyapunov functions for the special case of linear dynamical systems.

IV. DISCRETIZATION OF THE P-F OPERATOR

The purpose of this section is to review the set-oriented numerical methods for constructing finite-dimensional approximations of the P-F operator. The approximation arises as a Markov matrix defined with respect to a finite partition of the phase space.

A. Discretization – Markov matrix

In order to obtain a finite-dimensional (discrete) approximation of the continuous P-F operator, one considers a finite partition of the phase space X, denoted as

$$\mathscr{X} \doteq \{D_1, \cdots, D_L\}, \tag{30}$$

where $\cup_j D_j = X$. Such a partition may be constructed by taking quantization of states in $\mathscr{X}$. Instead of a Borel σ-algebra, consider now a σ-algebra of the all possible subsets of X_L. A real-valued measure μ_j is defined by ascribing to each element D_j a real number. Thus, one identifies the associated measure space with a finite-dimensional real vector space $\mathbb{R}^L$. The discrete P-F approximation arises as a matrix on this "measure space" $\mathbb{R}^L$.

For a mapping $T : X \to X$, the discrete approximation is constructed from its stochastic transition function $\delta_{T(x)}$. In particular, corresponding to a vector $\mu = (\mu_1, \cdots, \mu_L) \in \mathbb{R}^L$, define a measure on X as

$$d\mu(x) = \sum_{i=1}^{L} \mu_i \kappa_i(x) \frac{dm(x)}{m(D_i)}$$

where m is the Lebesgue measure and κ_j denotes the indicator function with support on D_j. The approximation, denoted by P, is now obtained as

$$\nu_j = \mathbb{P}[\mu](D_j) = \sum_{i=1}^{L} \int_{D_i} \delta_{T(x)}(D_j) \mu_i \frac{dm(x)}{m(D_i)} = \sum_{i=1}^{L} \mu_i P_{ij},$$

where

$$P_{ij} = \frac{m(T^{-1}(D_j) \cap D_i)}{m(D_i)}, \tag{31}$$

m being the Lebesgue measure. The resulting matrix is non-negative and because $T : D_i \to X$,

$$\sum_{j=1}^{L} P_{ij} = 1,$$

i.e., P is a Markov or a row-stochastic matrix.

Computationally, several short term trajectories are used to compute the individual entries P_{ij}. The mapping T is used to *transport* M "initial conditions" chosen to be uniformly distributed within a set D_i. The entry P_{ij} is then approximated by the fraction of initial conditions that are in the box D_j after one iterate of the mapping. In the remainder of the paper, the notation of this section is used whereby P represents the finite-dimensional Markov matrix corresponding to the infinite dimensional P-F operator $\mathbb{P}$.

B. Attractor sets & Invariant measures

The finite-dimensional Markov matrix P is used to numerically study the approximate asymptotic dynamics of the Dynamical system T; cf., [30], [20]. Recent research interest has focussed on carrying out *spectral analysis* of the Markov matrix to obtain statistical information on the asymptotic dynamics; cf. [16], [17], [18]. In particular, suppose $\mu \geq 0$ is an invariant probability measure (vector), i.e.,

$$\mu P = 1 \cdot \mu,$$

such that $\sum \mu_i = 1$ then the support of μ gives an outer approximation of the attractor and $\mu_i = \mu(D_i)$ measures the "weight" of the component D_i in attractor A [31]. The analysis has also been extended to interpret other portions of the Markov matrix's spectrum. In particular, dynamically relevant "almost invariant sets" correspond to eigenmeasures with eigenvalues close to unity [32]. The cyclic behavior within a attractor can be extracted by considering the complex unitary spectrum of the Markov chain [16], [17].

C. Example

In this example, a Markov matrix is constructed for the logistic map in a parameter regime where the solution shows chaotic behavior. The logistic map given by

$$T(x;\lambda) = \lambda x - x^3,$$

and is well-studied in the Dynamical Systems literature. Figure 1 depicts the spectrum of the P-F operator for $\lambda = \frac{3}{2}\sqrt{3} + 10^{-2}$ together with the invariant measure. As expected, the invariant measure captures the asymptotic behavior of trajectories of the logistic map. The peaks at the two ends and in the middle suggest that the trajectories on an average spend most of their time there. In addition to the unity eigenvalue, there is another eigenvalue very close to unity. This eigenvalue corresponds to the fact that there are two "almost invariant sets" embedded in the attractor.

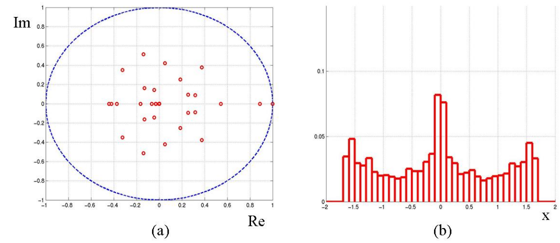

Fig. 1. (a) Eigenvalues and (b) the invariant measure of the discretized P-F matrix for the logistic map

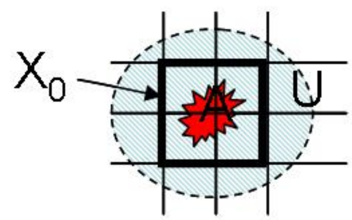

Fig. 2. A schematic of the three sets $A \subset X_0 \subset U$: A denotes the attractor set, X_0 is the support of its invariant measure approximation, and U is some neighborhood. The finite partition is shown as the rectangular grid in the background.

V. STABILITY IN FINITE-DIMENSION

In this section, discretization methods are used to approximate the Lyapunov measure. The existence of an approximation is related to yet weaker notions of stability, termed as coarse stability.

A. Matrix decomposition

We begin by presenting a decomposition result for the approximation P corresponding to a finite partition. This decomposition is a finite-dimensional analogue of Eq. (8). It is assumed that an approximation μ_0, to the invariant measure μ supported on the attractor set $A \subset X$, has been computed by evaluating a fixed-point the matrix P. An indexing is chosen such that the two non-empty complementary partitions

$$\mathscr{X}_0 = \{D_1, ..., D_K\}, \tag{32}$$
$$\mathscr{X}_1 = \{D_{K+1}, ..., D_L\} \tag{33}$$

with domains $X_0 = \cup_{j=1}^{K} D_j$ and $X_1 = \cup_{j=K+1}^{L} D_j$ distinguish the approximation of the attractor set from its complement set respectively. In particular, $A \subset X_0$, μ_0 is supported and non-zero on $\mathscr{X}_0$, and one is interested in stability with respect to the initial conditions in the complement X_1. For an attractor A with an invariant measure defined with respect to a neighborhood $U \supset A$, such sets exist for a sufficiently fine partition such that $A \subset X_0 \subset U$; cf., Figure 2. The following Lemma summarizes the matrix decomposition result.

Lemma 21 *Let P denote the Markov matrix for the mapping T in Eq. (1) defined with respect to the finite partition $\mathscr{X}$ in Eq. (30). Let $M \cong \mathbb{R}^L$ denote the associated measure space and μ denote a given invariant vector of P. Suppose $\mathscr{X}_0$ and $\mathscr{X}_1$ are the two* non-empty *components as in Eq. (32)-(33) defined with respect to μ such that $\mu > 0$ on $\mathscr{X}_0$; $\mu_i > 0$*

iff $D_i \in \mathscr{X}_0$. Let $M_0 \cong \mathbb{R}^K$ and $M_1 \cong \mathbb{R}^{L-K}$ be the measure spaces associated with $\mathscr{X}_0$ and $\mathscr{X}_1$ respectively. Then for the splitting $M = M_0 \oplus M_1$, the P matrix has a lower triangular representation

$$P = \begin{bmatrix} P_0 & 0 \\ \times & P_1 \end{bmatrix} \tag{34}$$

where $P_0 : M_0 \to M_0$ is the Markov matrix with row sum equal to one and $P_1 : M_1 \to M_1$ is the sub-Markov matrix with row sum less than or equal to one.

Proof: Use the splitting $M = M_0 \oplus M_1$ to express the invariant vector $\mu = [\mu^0, \mu^1]$ where $\mu^0 \in M_0$ and $\mu^1 \in M_1$. By construction, $\mu^0 > 0$ for all entries and $\mu^1 = 0$. Again, use the splitting to write

$$P = \begin{bmatrix} P_0 & P_2 \\ \times & P_1 \end{bmatrix}.$$

In order to prove the result, note that P is non-negative matrix such that

$$[\mu^0, 0] = \mu P = [\mu^0 P_0, \mu^0 P_2].$$

Since, $\mu^0 > 0$ so $P_2 = 0$. ■

We remark that this decomposition result does not explicitly require either the existence of the set U or any property $A \subset X_0 \subset U$ regarding the partition $\mathscr{X}_0$. These two however ensure that a) $\mathscr{X}_0$ and $\mathscr{X}_1$ are non-empty and b) the invariant vector is a good approximation of the invariant measure and hence the underlying attractor.

Example 22

1) Suppose x_0 is a locally stable fixed point of Eq. (1). The invariant measure is the Dirac delta measure supported on x_0, denoted by δ_{x_0}. Next, assume a partition such that $D_1 \subset U$, where U lies is the domain of attraction of x_0. The discrete approximation of the invariant measure is then given by

$$\mu_1 = 1, \qquad \mu_i = 0 \quad \text{for } i \neq 1.$$

where μ_i is the measure on cell D_i. The P matrix is given by

$$P = \begin{bmatrix} P_0 & P_2 \\ \times & P_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \times & P_1 \end{bmatrix}.$$

2) Consider next a locally stable period-two orbit $A = \{x_0, x_1\} \subset U$, a neighborhood in its domain of attraction. The physical measure is given by $\mu = \frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2}$. Assume a fine enough partition with $X_0 = D_1 \cup D_2$ such that $x_1 \in D_1$, $x_2 \in D_2$, $X_0 \subset U$, $T : D_1 \to D_2$, and $T : D_2 \to D_1$. It follows that the P matrix is given by

$$P = \begin{bmatrix} P_0 & P_2 \\ \times & P_1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \times & P_1 \end{bmatrix}$$

Our strategy is to study the stability in terms of properties of the matrix P_1 and define coarser (weaker) notions of stability with respect to initial conditions corresponding to this.

B. Coarse stability

In Sec. III, stability in continuous settings was shown to be related to the transience of the operator $\mathbb{P}_1$. In discrete settings, the stability is expressed in terms of the transient property of the stochastic matrix P_1.

Definition 23 (Transient states) *A sub-Markov matrix P_1 has only* **transient states** *if $P_1^n \to 0$, element-wise, as $n \to \infty$.*

Intuitively, it makes sense that if the invariant set A is stable or a.e. stable then the sub-Markov matrix P_1 is transient. Conversely, transience of P_1 is shown to imply yet weaker forms of stability referred to as **coarse stability** in this paper.

Definition 24 (Coarse Stability) *Consider an attractor $A \subset X_0$ together with a finite partition $\mathscr{X}_1$ of the complement set $X_1 = X \setminus X_0$. A is said to be* **coarse stable** *with respect to the initial conditions in X_1 if for an attractor set $B \subset U \subset X_1$, there exists no sub-partition $\mathscr{S} = \{D_{s_1}, D_{s_2}, \ldots, D_{s_l}\}$ in $\mathscr{X}_1$ with domain $S = \cup_{k=1}^{l} D_{s_k}$ such that $B \subset S \subset U$ and $T(S) \subseteq S$.*

For *typical* partitions, coarse stability means stability modulo attractor sets B with domain of attraction U smaller than the size of cells within the partition. In the infinite-dimensional limit, where the cell size (measure) goes to zero, one obtains stability modulo attractor sets with measure 0 domain of attraction, i.e., a.e. stability. Figure 3 compares some of the possibilities with a.e. stability in infinite-dimensional settings and coarse stability using finite partitions. The part (a) shows that measure 0 invariant sets such as unstable equilibrium (denoted by o) or a (dashed) line in the plane may arise in the complement X_1 even with a.e. stability. However, stable equilibrium with a domain of attraction of positive measure is ruled out. The parts (b) and (c) consider coarse stability in discrete settings with a rectangular partition in the background. The part (b) shows that a stable equilibrium (denoted by x) or an elongated attractor set with a smaller, than cell size, domain of attraction is possible with coarse stability. However, an attractor whose domain of attraction contains a sub-partition

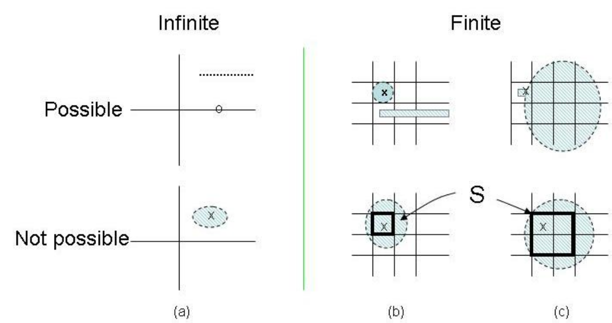

Fig. 3. A schematic comparing a.e. stability in infinite-dimensional setting (part (a)) to the coarse stability with finite partitions (part (b) and (c)). In either case, appropriate notion of stochastic stability is assumed ($\mathbb{P}_1$ and P_1 transient).

S (marked with bold lines in the Fig. 3) in the complement set is not possible. In particular, coarse stability rules out the case where the cell containing a stable equilibrium itself lies in its domain of attraction. The part (c) shows that it is possible to construct a partition where coarse stability holds, yet the domain of attraction is very large with respect to the partition. This is because the cell containing the stable equilibrium is not itself contained in the domain of its attraction. We believe this to be atypical for reasonable choices of *fine enough* finite partition with the lower figure in part (c) being a better representative. Nevertheless, the scale of partition is important in deducing stability as seen in the following example.

Example 25 Consider a scalar dynamical system

$$x_{n+1} = x_n - (x_n - a_1)(x_n - b)(x_n - a_2) \quad \text{for } x \in X \doteq [0,1], \tag{35}$$

where $0 < a_1 < \frac{1}{2} < b < a_2 < 1$; a_1, a_2 are stable and b is unstable. Consider a coarse partition

$$\mathscr{X} = \{[0,\frac{1}{2}],[\frac{1}{2},1]\}, \quad \mathscr{X}_0 = \{[0,\frac{1}{2}]\}, \quad \mathscr{X}_1 = \{[\frac{1}{2},1]\}$$

for which the Markov matrix arises as

$$P = \begin{bmatrix} 1 & 0 \\ 1-p & p \end{bmatrix}$$

for some $p < 1$. Hence, $P_1 = p < 1$ in this case is transient. Using the following theorem 26, this leads to coarse stability. The coarse stability thus misses the stable fixed point a_2 in the complement set $X_1 = [1/2,1]$. Next, consider any finite refinement of the partition $\mathscr{X}_1$. It is easy to verify that by choosing $a_2 - b$ to be sufficiently small, one again has the situation where P_1 on $\mathscr{X}_1$ is transient. However, for any given $b - a_2$, there exists a partition $\mathscr{X}_1$ that is fine enough so that b and a_2 lie within separate cells. For such a partition and its refinements, the Markov matrix P_1 will not be transient. In fact, the invariant measure's approximation supported on the cell containing a_2 will be persistent.

The theorem below formally links the transience of matrix P_1 to various notions of stability considered in this paper.

Theorem 26 *Assume the notation of the Lemma 21. In particular, A is an attractor set in $X_0 \subset X$ with approximate invariant measure supported on the finite partition $\mathscr{X}_0$ of X_0. $\mathbb{P}_1$ is the sub-Markov operator on $\mathscr{M}(A^c)$. P_1 is its finite-dimensional sub-Markov matrix approximation obtained with respect to the partition $\mathscr{X}_1$ of the complement set $X_1 = X \setminus X_0$. For this*

1) *Suppose a Lyapunov measure $\bar{\mu}$ exists such that*

$$\mathbb{P}_1\bar{\mu}(B) < \bar{\mu}(B) \tag{36}$$

for all $B \subset \mathscr{B}(X_1)$, and additionally $\bar{\mu} \approx m$, the Lebesgue measure. Then the finite-dimensional approximation P_1 is transient.

2) *Suppose P_1 is transient then A is coarse stable with respect to the initial conditions in X_1.*

Proof: Before stating the proof, we claim that for any two sets S_1 and S such that $S_1 \subset S$, if $\mu \approx m$ then

$$\mu(S_1) = \mu(S) \quad \text{if and only if} \quad m(S_1) = m(S) \tag{37}$$

Denote $S_1^c := S \setminus S_1$ to be the complement set. We have, $\mu(S_1) = \mu(S)$ implies $\mu(S_1^c) = 0$ which in turn implies $m(S_1^c) = 0$ and thus $m(S_1) = m(S)$.

1. We first present a proof for the simplest case where the partition $\mathscr{X}_1$ consists of precisely one cell, i.e., $\mathscr{X}_1 = \{D_L\}$. In this case, $P_1 \in [0,1]$ is a scalar given by

$$P_1 = \frac{m(T^{-1}(D_L) \cap D_L)}{m(D_L)}, \tag{38}$$

where m is the Lebesgue measure. We need to show that $P_1 < 1$. Denote,

$$S = \{D_L\}, \quad S_1 = \{x \in D_L : T(x) \in D_L\}. \tag{39}$$

Clearly, $S_1 \subset S$ and existence of Lyapunov measure $\bar{\mu}$ satisfying Eq. (36) implies that

$$\bar{\mu}(S_1) = \mathbb{P}_1\bar{\mu}(S) < \bar{\mu}(S).$$

Using (37), $m(S_1) \neq m(S)$ and since $S_1 \subset S$, we have $m(S_1) < m(S)$. Using Eqs. (38) and (39), this implies $P_1 < 1$, i.e., P_1 is transient.

We prove the result for the general case, where $\mathscr{X}_1$ is a finite partition, by contradiction. Suppose P_1 is not transient. Then using either the following Theorem 28, or a general result from the theory of finite Markov chains [24], [33], there exists atleast one non-negative invariant probability vector ν such that

$$\nu \cdot P_1 = \nu. \tag{40}$$

Let,

$$S = \{x \in D_i : \nu_i > 0\}, \quad S_1 = \{x \in S : T(x) \in S\}.$$

It is claimed that

$$m(S_1) = m(S). \tag{41}$$

We first assume the claim to be true and show the desired contradiction. Clearly, $S_1 \subset S$ and if the claim were true, (37) shows that

$$\bar{\mu}(S_1) = \bar{\mu}(S). \tag{42}$$

Next, because $S \subset X_1$,

$$\mathbb{P}_1\bar{\mu}(S) = \bar{\mu}(T^{-1}(S) \cap X_1) \geq \bar{\mu}(T^{-1}(S) \cap S).$$

and this together with Eq. (42) gives

$$\mathbb{P}_1\bar{\mu}(S) \geq \bar{\mu}(S)$$

for a set S with positive Lebesgue measure. This contradicts Eq. (36) and proves the theorem.

It remains to show the claim. Let $\{i_k\}_{k=1}^l$ be the indices with $\nu_{i_k} > 0$. Eq. (40) gives

$$\sum_{k=1}^{l} \nu_{i_k} [P_1]_{i_k j_m} = \nu_{j_m} \quad \text{for} \quad m = 1, \ldots, l.$$

Taking a summation $\sum_{m=1}^{l}$ on either side gives

$$\sum_{k=1}^{l} \nu_{i_k} \sum_{m=1}^{l} [P_1]_{i_k j_m} = 1.$$

Since, individual entries are non-negative and ν is a probability vector, this implies

$$\sum_{m=1}^{l}[P_1]_{i_k j_m} = 1 \;\; k = 1,\ldots,l,$$

i.e., the row sums are 1. Using formula (31) for the individual matrix entries, this gives

$$\sum_{m=1}^{l} m(T^{-1}(D_{j_m})\cap D_{i_k}) = m(D_{i_k}),$$
$$\text{therefore, } m(T^{-1}(\cup_{m=1}^{l} D_{j_m})\cap D_{i_k}) = m(D_{i_k}) \quad k = 1,\ldots,l,$$

where we have used the fact that the pre-image sets are disjoint and $\cup T^{-1}(D_{j_m}) = T^{-1}(\cup D_{j_m})$. However, by construction $S = \cup_{m=1}^{l} D_{j_m}$ and thus

$$m(T^{-1}(S)\cap D_{i_k}) = m(D_{i_k}) \;\text{ for } k = 1,\ldots,l.$$

Taking a summation $\sum_{k=1}^{l}$ on either side gives

$$m(T^{-1}(S)\cap S) = m(S),$$

precisely as claimed in Eq. (41). This completes the proof for the general case.

2. Suppose P_1 is transient. To show that A is coarse stable, we proceed by contradiction. Indeed, using definition 24, if A was not coarse stable then there exists an attractor set $B \subset U \subset X_1$ with a sub-partition $\mathscr{S} = \{D_{s_1},...,D_{s_l}\}$, $S = \cup_{k=1}^{l} D_{s_k}$ such that $B \subset S \subset U$ and $T(S) \subseteq S$. Since, the set S is left invariant by mapping T,

$$P_{s_k j} = \frac{m(T^{-1}(D_j)\cap D_{s_k})}{m(D_{s_k})} = 0,$$

whenever $D_j \notin \mathscr{S}$. Moreover, because $T : S \to S$,

$$\sum_{j=1}^{l}[P_1]_{s_i s_j} = 1 \qquad i = 1,...,l,$$

i.e., P_1 is a Markov matrix with respect to the finite partition $\mathscr{S}$. Form the general theory of Markov matrix [24], there then exists an invariant probability vector ν such that

$$\nu \cdot P_1^n = \nu,$$

for all $n > 0$, and P_1 is not transient. ■

Corollary 27 *Consider $T : X \to X$ in Eq. (1) with an invariant set $A \subset U(\varepsilon) \subset X_0 \subset X$, $U(\varepsilon)$ is some ε-neighborhood of A. P_1 is the sub-Markov matrix with respect to a finite partition of the complement set $X_1 = X \setminus X_0$. Suppose A is stable a.e. with geometric decay with respect to some finite measure $m \in \mathscr{M}(X \setminus U(\varepsilon))$. Then, P_1 is transient.*

Proof: Theorem 18 shows that an equivalent Lyapunov measure exists whenever A is a.e. stable with geometric decay. The result follows from part 1 of the Theorem 26 above. ■

In summary, a.e. stability implies P_1 is transient, while one can only conclude a weaker coarse stability given transience of P_1.

C. Formulae for Lyapunov measure

There are a number of equivalent characterizations of the transience, expressed in Definition 23, of the sub-Markov matrix P_1. These are summarized in the theorem below and will be used to obtain computational algorithms for deducing coarse stability.

Theorem 28 *Suppose P_1 denotes a sub-Markov matrix. Then the following are equivalent*

1) *P_1 is transient,*
2) *$\rho(P_1) \le \alpha < 1$,*
3) *the infinite-series $I + P_1 + P_1^2 + \ldots$ converges,*
4) *there exists a Lyapunov measure $\bar{\mu} > 0$ such that $\bar{\mu}P_1 \le \alpha\bar{\mu}$ where $\alpha < 1$.*

Proof: ($1 \Longrightarrow 2$) Since P_1 is assumed to be a sub-Markov matrix, $\rho(P_1) \le 1$. By non-negativity of P_1, $\rho(P_1)$ is in fact an eigenvalue of P_1 with a non-negative vector; cf., Sec 8.3 in [33]. As a result, if $\rho(P_1) = 1$ then there exists $\nu \ge 0$, $\nu \ne 0$ such that

$$\nu P^n = \nu$$

for all n. This contradicts 1.
($2 \Longrightarrow 3$) With $\rho(P_1) < 1$, the inverse $(I - P_1)^{-1}$ exists and is in fact analytic with the series expansion

$$(I - P_1)^{-1} = I + P_1 + P_1^2 + \ldots \tag{43}$$

In particular, the series converges.
($3 \Longrightarrow 4$) Choose $m > 0$, and set

$$\bar{\mu} = m\cdot(I - P_1)^{-1} = m + mP_1 + mP_1^2 + \ldots$$

The non-negativity of P_1 together with convergence of series implies that the inverse $(I - P_1)^{-1}$ is itself a non-negative matrix [34]. As a result, $\bar{\mu} > 0$ for $m > 0$. A simple calculation then shows that

$$\bar{\mu}\cdot P_1 - \bar{\mu} = -m < 0.$$

Because of the strict inequality, there must then exist an $\alpha < 1$ such that

$$\bar{\mu}\cdot P_1 \le \alpha\bar{\mu}.$$

($4 \Longrightarrow 1$) By taking repeated powers, $\bar{\mu}\cdot P_1^n < \alpha^n\bar{\mu}$. The right hand side converges to zero. Since P_1 is a non-negative matrix and $\bar{\mu} > 0$, this implies that $P_1^n \to 0$ as $n \to \infty$. ■
If it exists, an approximation of the Lyapunov measure can be computed as a solution to a system of linear inequalities

$$\bar{\mu}\cdot(\alpha I - P_1) \;>\; 0, \tag{44}$$
$$\bar{\mu} \;>\; 0. \tag{45}$$

Such a solution is efficiently computed using Linear Programming (LP) methods. For a given $m > 0$, convergence of the infinite-series in Eq. (43) provides for another method for computing the approximation:

$$\bar{\mu} = m\cdot(I - P_1)^{-1} = m + m\cdot P_1 + m\cdot P_1^2 + \ldots \tag{46}$$

In summary, transience of the Markov chain P_1 can be expressed in three equivalent ways useful for distinct computational approaches:

1) Verify a spectral condition $\rho(P_1) \leq \alpha < 1$,
2) Compute a Lyapunov measure $\bar{\mu}$ using a series formulation as in Eqs. (46),
3) Compute a Lyapunov measure using Linear programming as in Eqs. (44)-(45).

The parallels with the linear dynamical system are summarized in the Table 1. The spectral condition is a counterpart of $\rho(A) < 1$ for the linear dynamical system. The series expansion corresponds to the series solution of the Lyapunov equation. It can also be obtained as a solution of a linear equation. Finally, the linear programming based formulation arises due to the non-negativity of the matrix P_1. It does not share any obvious counterpart in the linear setting.

TABLE I
CONDITIONS FOR RECURRENCE AND TRANSIENCE

	Linear (A)	Nonlinear (P_0, P_1)
Invariant set	$0 = A \cdot 0$	$\mu = \mu \cdot P_0$
Spectral condition	$\rho(A) < 1$	$\rho(P_1) < 1$
Series-expansion	$A^T \cdot P \cdot A - P = -Q$	$\bar{\mu} = m \cdot (I - P_1)^{-1}$
Linear inequalities	$--$	$\bar{\mu} \cdot P_1 < \bar{\mu}$

Remark 29 Computationally, it is most attractive to verify stability using the linear inequalities (44)-(45). We used the MATLAB command *linprog* to verify stability in the example problems described in the following section. One important point to note is that the inequality (44) needs to be strict for deducing stability. As a result, the inequalities (44)-(45) are implemented in MATLAB as

$$\bar{\mu} \cdot P_1 \quad \leq \quad \alpha \bar{\mu} - \varepsilon, \tag{47}$$
$$\bar{\mu} \quad \geq \quad 0, \tag{48}$$

where ε is a small positive constant used to enforce strict inequality and $\alpha \leq 1$.

The Lyapunov measure and the computational framework is expected to be particularly useful for control design with the objective of stabilization of an equilibrium or an invariant set. This framework, however, is different from the Lyapunov *function* based computational methods that have appeared in recent literature. In contrast to the set-wise measure theoretic stability concepts of this paper, the SOS polynomial based papers [11], or set-oriented papers [35], or papers utilizing dynamic programming and numerical approximation ideas for optimal control [36] all aim to synthesize point-wise functions: density, approximate Lyapunov function, or optimal value functions, respectively. We will establish more concrete connection between optimal control and Lyapunov measure in a separate publication focussing on control.

D. Examples

Example 30 Consider dynamics on a finite set,

$$\begin{aligned} T(x_i) &= x_0, \quad \text{for } i = \{0,1\} \\ T(y_i) &= x_1, \quad \text{for } i = \{1,\ldots,N\}, \end{aligned} \tag{49}$$

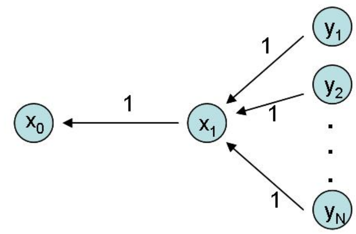

Fig. 4. A schematic of the discrete dynamics in Eq. 49.

TABLE II
LYAPUNOV FUNCTION V AND MEASURE $\bar{\mu}$ FOR THE DISCRETE DYNAMICS IN EQ. 49

Complement set	x_1	y_i
V	$\frac{1}{2}$	1
$\bar{\mu}$	$N+1$	1

as shown in Fig. 4. The state $\{x_0\}$ is a globally stable attractor. Table 2 gives a Lyapunov function and measure on the complement set $\{x_1, y_1, \ldots, y_N\}$. The large value of Lyapunov measure $\bar{\mu}$ at the point x_1 is a reflection of the size (N) of its pre-image set. In regions (cells) such as these, where the flow is squeezed through a narrow region, the Lyapunov measure will have a high value. Due to the dual nature of Lyapunov measure and Lyapunov function the behavior of Lyapunov measure and Lyapunov function is exactly opposite. Lyapunov measure takes smaller value on the sets which are away from the invariant set and larger value on the set which are closer to the invariant set, Lyapunov function on the other hand takes lower value on the states which are closer to the equilibrium point and larger value on the states which are further away from the equilibrium point.

Example 31 Consider the 1-d cubic logistic map

$$x_{n+1} = \lambda x_n - x_n^3, \tag{50}$$

with $\lambda = 2.3$ and $X = [-1.5, 1.5]$. The value of λ is chosen to be at the "edge," where a sequence of period-doubling bifurcations lead to chaos. Figure 5 (a) shows the asymptotic attractor sets obtained as a function of initial conditions in X. There are two symmetric attractors, that are stable in the sense that any typical initial condition asymptotes too one of these sets. Figure 5 (b) verifies this with the aid of the Lyapunov measure on the complement set to the support of the two invariant measures. We refer the reader to Sec. IV for details on set-oriented approximation of the P-F operator. The Lyapunov measure was computed as a solution of the linear inequalities Eqs. (47)-(48). Linear programming (MATLAB command linprog) was used to obtain this solution. The invariant measures (in red) correspond to the two attractors and the Lyapunov measure (in blue) is computed on the complement set. We remark that one does not have global stability, for initial conditions in X, for either of the attractors. However, existence of a Lyapunov measure ensures that in a coarse sense, any initial condition in the complement set asymptotes to the support of one of the two invariant measures.

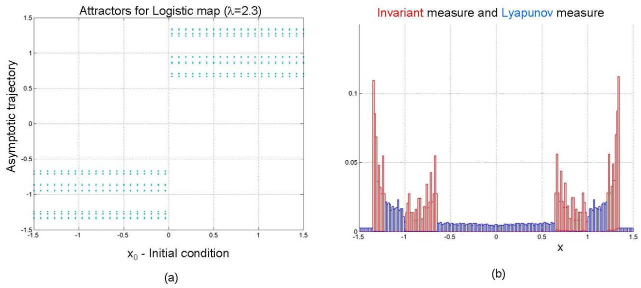

Fig. 5. Asymptotic behavior of the logistic map in Eq. (50): (a) attractor sets as a function of initial condition x_0 and (b) the invariant measures for these attractor sets (in red) and the Lyapunov measure (in blue) verifying their stability.

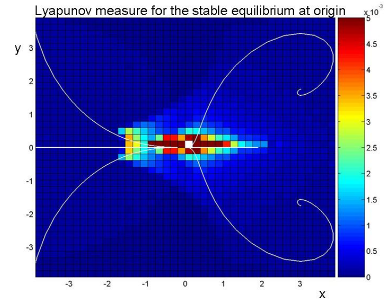

Fig. 7. Lyapunov measure for the equilibrium at origin for ODE in Eq. (52) on the glued domain X. The invariant measure is supported on single cell shown in white at the origin. White curves denote some representative trajectories.

Example 32 Consider the ODE for the Vanderpol oscillator

$$\ddot{x} - (1-x^2)\dot{x} + x = 0. \tag{51}$$

A dynamical system T is obtained after numerical integration of the ODE over a time-interval of $\Delta t = 1$. A suitably large Δt is chosen so $T : X \to X$, where $X = [-3,3] \times [-3,3]$ is a finite box containing the unstable origin and the globally stable Vanderpol limit cycle. Figure 6 (a) depicts the approximation of the invariant measure corresponding to this limit cycle and part (b) shows its Lyapunov measure. In the region inside the limit cycle, the measure shows moderate variations with larger values near the limit cycle. Outside the limit cycle, there are two sharp peaks denoting the regions where most trajectories in the phase space squeeze through before converging uniformly to the vicinity of the limit cycle. The figure shows some of these trajectories (in white) together with the peaks (denoted as "max") in the value of the Lyapunov measure.

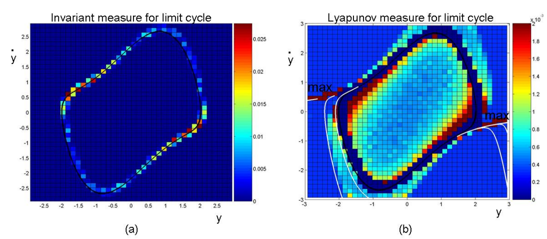

Fig. 6. (a) Invariant measure (b) Lyapunov measure for the Vanderpol oscillator in Eq. (51). The limit cycle is shown as a black curve and white curves denote some representative trajectories. For the Lyapunov measure, the maximum value of 0.026 was seen at the two regions denoted as "max." The color axis in part (b) is limited to $[0, 0.002]$ to better represent the variations in the value of Lyapunov measure.

Example 33 We next consider a dynamical system T corresponding to the ODE

$$\begin{aligned} \dot{x} &= -2x + x^2 - y^2, \\ \dot{y} &= -6y + 2xy, \end{aligned} \tag{52}$$

In [3], the origin was shown to be a.e. stable with respect to initial conditions in $\mathbb{R}^2$. This example does not have any compact T-invariant set X that contains all of its equilibria. The trajectory for any initial condition on x-axis with $x > 2$ grows unbounded. To apply the results of this paper, we consider the domain to be $X = [-4,4] \times [-4,4]$ and glue its boundaries. In particular, the left boundary $(x = -4, y)$ is glued to the right boundary at $(x = 4, y)$, the upper boundary $(x, y = 4)$ with $x > 0$ is glued to $(-x, y = 4)$, and similarly on the lower boundary $y = -4$. Inside the glued domain, the dynamics are described by the ODE in Eq. (52). The dynamical system for the same was constructed using numerical integration with $\Delta t = 0.2$. Figure 7 depicts the Lyapunov measure on the complement (to the origin) set verifying coarse stability of the origin in X. Also shown are typical trajectories showing the convergence to the origin. The peaks in the Lyapunov measure are consistent with the convergence of typical trajectories, a few of which are shown in white.

E. Duality - Lyapunov function

In this section, we consider the discrete counterpart of the Lyapunov function. In continuous settings, the analysis in Sec. III-C and in particular, Eq. (24) shows that Lyapunov function is related to the dual of the P-F operator. In discrete settings, one way to proceed is to consider the transpose of the matrix P_1. Indeed, the discrete analogue of Eq. (24) is given by

$$(I - P_1)V = g, \tag{53}$$

where multiplication on the right is equivalent to taking a transpose of P_1 (and multiplying on left), and g is a positive vector on the partition $\mathscr{X}_1$. If P_1 is transient then using the results of Theorem 28, a unique and positive solution V exists for any positive g. However, unlike the infinite-dimensional case, V is in general not a Lyapunov function except for a special case where P_1 is additionally deterministic.

Definition 34 (Deterministic Markov matrix [13]) *A Markov or a sub-Markov matrix P_1 is* **deterministic** *if the individual entries are either* 0 *or* 1.

It easily follows that for any row of a deterministic P_1, at most one entry is non-zero. It is necessarily 1 for a Markov matrix but may be 0 for a sub-Markov matrix. The interpretation here is that if $P_{1\,ij}$ is 1, then *almost all* the states in the i^{th} cell go to the j^{th} cell after one iterate of the mapping T. If $P_{1\,ij} = 0$ for all j, then the states in i^{th} cell are transient in 1-step. Since all of the states within a cell behave identically, it is possible to set one value for the Lyapunov function over the cell. Said another way, the indicator functions κ_i are the basis of the Lyapunov function with co-ordinate V_i, i.e.,

$$V(x) = \sum_i V_i \kappa_i(x), \tag{54}$$

where κ_i is the indicator function for cell D_i. Analogously, define

$$g(x) = \sum_i g_i \kappa_i(x),$$

The following theorem then shows that the solution V to Eq. (53) in fact gives the Lyapunov function.

Theorem 35 *Consider a mapping $T : X \to X$ with an attractor A, and a sub-Markov and deterministic matrix P_1 that is defined for a finite partition of the complement set. Assume P_1 is transient and let V be a solution of Eq. (53) for a given positive g. Then $V(x)$ defined by Eq. (54) is a Lyapunov function with*

$$V(x) - V(Tx) = g(x),$$

for all $x \in X_1$ with $Tx \in X_1$. $V(x) = g(x)$ where $Tx \in X_0$.

Proof: By transience of P_1, a unique positive solution V exists. If states in the cell i go to cell j in one iterate of mapping T then

$$(P_1 V)_i = V_j.$$

Hence, the co-ordinate form of the Eq. (53) reads

$$V_i - V_j = g_i \tag{55}$$

For x in cell i with Tx in cell j,

$$V(x) = V_i, \quad V(Tx) = V_j, \quad g(x) = g_i.$$

Using Eq. (55),

$$V(x) - V(Tx) = g(x), \tag{56}$$

for x in cell i. Since i is arbitrary the result follows for all $x \in X_1$ such that $Tx \in X_1$. If $Tx \in X_0$, the states in cell i are transient in 1 step, $(P_1 V)_i = 0$, and $V(x) = g(x)$ using very similar arguments. For a given $g > 0$, V is then a Lyapunov function by Eq. (56). ∎

For the deterministic case, one can use a Lyapunov function V to obtain a Lyapunov measure $\bar{\mu}$ and vice-versa under one additional assumption on P_1. We say that a Markov or a sub-Markov matrix P_1 is **1-1** if P_1 is deterministic and has atmost one non-zero entry in each column. For such a P_1, set

$$\bar{\mu}_i = \frac{1}{V_i}. \tag{57}$$

Now, if $V_i > 0$ is a discrete Lyapunov function so $V_j < V_i$ whenever $P_{1\,ij} = 1$, one has

$$(\bar{\mu} P_1)_j = \bar{\mu}_i = \frac{1}{V_i} < \frac{1}{V_j} = \bar{\mu}_j,$$

i.e.,

$$\bar{\mu} P_1 < \bar{\mu},$$

and $\bar{\mu}$ is a Lyapunov measure. The converse follows similarly. In fact, the inverse relationship in Eq. (57) can be further generalized. Let, $h(\cdot)$ be any monotonically decreasing positive function of its argument then $\bar{\mu} = h(V)$ is a Lyapunov measure for a given V and $V = h(\bar{\mu})$ is a Lyapunov function for a given $\bar{\mu}$. In the following section, we extend this relationship to continuous settings.

VI. Relation between Lyapunov measures and functions

Under certain conditions, it is also possible to relate the Lyapunov function and the Lyapunov measure for the infinite-dimensional case. The motivation here is derived from the relationship in Eq. (57) for the discrete case and the results in Section 3 of [3], where the relationship between density function and Lyapunov function is given.

In this section, we impose an additional assumption of C^1-invertibility (diffeomorphism) on the mapping $T : X \to X$ in Eq. (1). For the diffeomorphism T, define

$$J^{-1}(x) = \left|\frac{dT^{-1}}{dx}(x)\right|$$

where $|\cdot|$ denotes the determinant of the Jacobian $T^{-1}(x)$ as evaluated at x. Because $T \circ T^{-1}(x) = x$, $J(x) = |\frac{dT}{dx}(T^{-1}(x))|$. The real-valued function $J^{-1}(x)$ has a special significance because it gives the density of measure $\mathbb{P}[m]$ with respect to the Lebesgue measure m. In particular,

Lemma 36 *Let $\mathbb{P}$ denote the P-F operator for the mapping $T : X \to X$ then*

$$d\mathbb{P}m(x) = J^{-1}(x)dm(x). \tag{58}$$

Next, suppose $f(x)$ denotes the density of an absolutely continuous measure μ with respect to m, i.e., $d\mu(x) = f(x)dm(x)$, then

$$d\mathbb{P}\mu(x) = f(T^{-1}(x))J^{-1}(x)dm(x). \tag{59}$$

Proof: Eq. (58) follows from

$$\begin{aligned}\mathbb{P}m(A) &= \int_X \chi_A(Tx)dm(x) = \int_X \chi_A(x)dm(T^{-1}x)\\ &= \int_X \chi_A(x)J^{-1}(x)dm(x).\end{aligned}$$

Eq. (59) follows from

$$\mathbb{P}\mu(A) = \int_X \chi_A(Tx)f(x)dm(x) = \int_X \chi_A(x)f(T^{-1}(x))J^{-1}(x)dm(x)$$

∎

A. Relationship

The purpose of this Section is to present the main result relating the Lyapunov measure and function under the additional assumption that $J(x) < 1$.

Theorem 37 *Let A be the invariant set for a dynamical system T and assume that $J(x) < 1$ for all $x \in A^c$. Then the following statements are true:*

1) *Suppose the invariant set A is a.e. stable with the Lyapunov measure $\bar{\mu}$ satisfying*

$$d\bar{\mu}(x) - d\mathbb{P}_1\bar{\mu}(x) = g(x)dm(x), \tag{60}$$

where $g(x) \geq 0$. Then

$$V(x) = \left(\frac{d\bar{\mu}}{dm}(x)\right)^{-1} \tag{61}$$

is a Lyapunov function with the property

$$V(x) < V(T^{-1}(x)). \tag{62}$$

2) *Suppose the invariant set A is stable with Lyapunov function V satisfying*

$$J^{-1}(x)V(x) < V(T^{-1}(x)) \tag{63}$$

then the measure

$$\bar{\mu}(B) = \int_B \frac{1}{V^\beta(x)} dm(x)$$

is a Lyapunov measure such that

$$\bar{\mu}(T^{-1}B) < \bar{\mu}(B) \tag{64}$$

for all $B \subset \mathscr{B}(A^c)$ with $m(B) > 0$. $\beta \geq 1$ is a suitable constant chosen so $\frac{1}{V^\beta}$ is integrable.

Proof:

1) Using Lemma 36,

$$d\bar{\mu}(x) - d\mathbb{P}_1[\bar{\mu}](x) = [V^{-1}(x) - V^{-1}(T^{-1}(x))J^{-1}(x)]dm(x).$$

Equation (60) then implies

$$V^{-1}(x) - V^{-1}(T^{-1}(x))J^{-1}(x) = g(x) \geq 0$$

and

$$V^{-1}(x) \geq V^{-1}(T^{-1}(x))J^{-1}(x) > V^{-1}(T^{-1}(x)).$$

This gives the desired result in Eq. (62).

2) Because $J^{-1}(x) > 1$ and $\beta \geq 1$,

$$J^{-1}(x)V(x) < V(T^{-1}(x)) \quad \text{which implies}$$
$$J^{-1}(x)V^\beta(x) < V^\beta(T^{-1}(x)),$$

i.e.,

$$\frac{J^{-1}(x)}{V^\beta(T^{-1}(x))} < \frac{1}{V^\beta(x)}$$

So for any positive Lebesgue measure set $B \subset \mathscr{B}(A^c)$,

$$\int_B \frac{J^{-1}(x)}{V^\beta(T^{-1}(x))} dm(x) < \int_B \frac{1}{V^\beta(x)} dm(x),$$

where β is a suitable constant that ensures that $\frac{1}{V^\beta(x)} \in \mathscr{L}^1(A^c)$. Now, set

$$\frac{d\bar{\mu}(x)}{dm(x)} = \frac{1}{V^\beta(x)}$$

and using Lemma 36, the above integral gives

$$\int_{T^{-1}(B)} d\bar{\mu}(x) < \int_B d\bar{\mu}(x)$$

The inequality in Eq. (64) follows.

∎

Note that on a transitory complement set A^c, the point $T(x)$ may lie in A and hence Tx may not be well-defined. However, $T^{-1}(x)$ is well-defined for all $x \in A^c$ and the Lyapunov function inequality is expressed in this form. Finally, we remark that for an ODE with vector-field $\underline{u}$ corresponding to a dynamical system T, the condition is $J(x) < 1$ if and only if $\nabla \cdot \underline{u} < 0$. The latter is indeed the assumption in [3], where the relationship between Lyapunov function and density function was first described.

VII. Discussion & Conclusions

In nonlinear control, Lyapunov functions have primarily been used for verifying stability and stabilization, using control, of an equilibrium solution. An equilibrium is only one of the many recurrent behavior that are possible in nonlinear dynamical systems. A stable periodic orbit is a simple example of non-equilibrium behavior but *strange*r *attractors* arise even in low-dimensions. For e.g., the Lorentz attractor and the chaotic attractor of the logistic map in Fig. 1. In higher dimensions such as distributed systems, non-equilibrium behavior is the norm.

In this paper, we argued that measure-theoretic stochastic approaches are a key to the study of non-equilibrium behavior in dynamical systems. Indeed, stochastic methods have come to be viewed as increasingly relevant for the study of global *recurrent* behavior such as attractor sets even in deterministic dynamical systems. Lyapunov measures, introduced in this paper, are a stochastic counterpart to the notion of *transience* and thus useful for verifying (weak forms of) stability of the recurrent attractor sets. Next, recent advances using set-oriented numerical approaches for the discretization of the stochastic operators have made the calculation of recurrent attractor sets as invariant measures routine. There are two ideas of interest here: a) non-equilibrium chaotic behavior is described more naturally on sets as opposed to with points, and b) a measure-theoretic description allows for a coarse and multi-scale study of such behavior. Either provide for reduction of complexity compared to a point-wise description. While, evolution of points is nonlinear and chaotic, the evolution of (measures supported on) sets is linear and well-behaved. In our paper, the discretization leads to coarser and multi-scale notions of stability which generalizes in a natural way the almost everywhere stability of [3].

It is noted that the presence of unstable points in the complement set is typically useful for the stabilization problem. The existence of point-wise positive Lyapunov function with everwhere notion of stability precludes such points. The a.e. notion of stability, first introduced in [3], allows for such points. It even allows for stable sets with Lebesgue measure 0 region of recurrence. The intuition being that such sets are not important from the point of view of any meaningful optimization criterion or that even smallest noise will in general destroy the recurrence. The coarse notions of the stability as a consequence of discretization carry this one step further. In effect, it allows for even typical stable recurrent sets with small (than the quantization size) regions of attraction. Once again, the intuition is that such sets are either not important for the given scale or that large enough (size of quantization) noise makes them irrelevant. We will investigate these ideas for the purposes of control design in a separate publication.

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