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Nonlinear Stabilization via Control-Lyapunov Measure

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Abstract— **This paper is concerned with computational methods for Lyapunov-based control design of an attractor set of a nonlinear dynamical system. Based upon a stochastic representation of deterministic dynamics, a Lyapunov measure is used for these purposes. This paper poses and solves the co-design problem of jointly obtaining the control Lyapunov measure and a controller. The computational framework is based upon a set-oriented numerical approach. Using this approach, the co-design problem leads to a finite number of linear inequalities whose solutions define the feasible set of stabilizing controllers. We provide a proof of existence for a stochastic version of such a controller while the deterministic restriction is posed as the solution of a related integer programming problem. Mathematical programming techniques may be employed to obtain such controllers. Finally, an example is provided to illustrate the ideas.**

I. Introduction

For nonlinear dynamical systems, Lyapunov function-based methods play a vital role in both stability analysis and control synthesis [1]. The synthesis typically relies on the *co-design* of the control Lyapunov function (CLF) and the controller. This is historically an important but challenging problem for the general class of nonlinear systems [2]. For linear systems, the co-design problem – after carrying out a suitable change of co-ordinate – reduces to a linear matrix inequality (LMI). These LMI define a feasible *convex* set in both the CLF and the control. In practice, semidefinite programming tools can be used to obtain a solution to the linear co-design problem [3]. For nonlinear systems, the problem of designing a Lyapunov based controller has historically relied on either using methods such as feedback linearization to construct a CLF [4] or simply *assuming* a CLF [5]. With a given CLF, the stabilizing control law is either obtained by using the Sontag's formula [6] or more generally using methods of nonlinear programming.

Spurred by the need for a more systematic and computationally attractive framework for the solution of this important problem, there have been a number of studies on exact and approximate approaches in recent years. A popular recent approach is to employ *convex optimization using SOS polynomial basis;* cf., [7]. The control synthesis is based upon using the Rantzer's density function [8] together with a polynomial basis for its numerical representation [9]. The resulting convex optimization problem can be solved using semidefinite programming. The other avenue is to synthesize *optimal control via efficient solutions of the HJB equation*; cf., [10], [11]. However, synthesis using dynamic programming approaches is in general complex; cf. [12]. More significantly, even though the solution to the HJB equation leads to a Lyapunov function, it may be an overkill if one is interested in simply obtaining a stabilizing control. Also, the non-unique nature of the Lyapunov function ought to lead to a *feasibility problem* as opposed to a single optimal control solution. Finally, certain *graph-theoretic approaches* have also been recently proposed for approximating Lyapunov functions with set-oriented description of dynamics; cf., [13]. These papers exploit the graph-theoretic nature of data structure implicit in set-oriented discretizations and propose the shortest-path Dijkstra algorithm [12] to construct an approximate solution to the HJB equation.

The objective of this paper is to use stochastic duality arguments for the purposes of a Lyapunov-based control design in deterministic nonlinear settings. Instead of a point-wise notion of the Lyapunov function, our work relies on weak and set-wise notions of stability. These notions were introduced in our earlier work [14] using the concept of **Lyapunov measure**. Lyapunov measure is a dual to Lyapunov function and is closely related to Rantzer's density function. Just as an invariant measure is a stochastic counterpart to the invariant set, the existence of Lyapunov measure gives a stochastic conclusion on the stability of the invariant measure. Using set-oriented numerical methods, the approximation/computation of Lyapunov measure can be cast as a solution to a finite system of linear inequalities.

In this paper, we extend this framework to the question of Lyapunov-based control design. There are three contributions of this paper. One, we develop a computational framework whereby set-oriented numerical methods are used to efficiently co-design Lyapunov measure and the control. In particular, the co-design problem is shown to yield a set of feasibility constraints expressed as linear inequalities. The control Lyapunov measure as well as the controller is efficiently computed using linear programming. Two, the construction leads to measure-theoretic and weaker notions of stabilization. This is related to the notion of *coarse stability* defined in [14], which is more natural vis-a-vis chaotic and nonlinear dynamics. This also brings to fore the important question of designing controllers that utilize beneficial aspects of dynamics. To expand on this point a bit, we quote from the Ott-Grebogi-Yorke (OGY) paper on controlling chaos [15]: *"Assuming the motion of the free-running [uncontrolled] chaotic orbit to be ergodic, eventually the*

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chaotic wandering of an orbit trajectory will bring it close to the chosen unstable periodic orbit or steady state. When this occurs, we can apply small controlling perturbations to direct the orbit to the desired periodic motion or steady state [16]." Using a simple example of the chaotic logistic map, we show that the computational method proposed in this paper can potentially provide a systematic framework for exploiting nonlinear dynamics for control and realizing the vision of this quote.

The outline of this paper is as follows. In Section II, preliminaries and the main results of [14] are reviewed. Section III sets up the control problem. The computational framework for the co-design problem appears in the Section IV. Certain existence results for the same are also discussed. Section V summarizes the results of an example and VI presents the conclusions.

II. Lyapunov measure & stability

In [14], we used stochastic duality arguments to investigate stability for an invariant set $A \subset X$ of a dynamical system $T : X \subset \mathbb{R}^n \to X$. T is assumed to be a continuous and non-singular map and $X \subset \mathbb{R}^n$ is a compact set. $\mathscr{B}(X)$ denotes the Borel σ-algebra on X and $\mathscr{M}(X)$ the vector space of bounded real-valued measures on $\mathscr{B}(X)$.

The basic idea was to define certain measure theoretic notions of stability. One such definition appears below:

Definition 1 (Almost everywhere stable): An attractor set A for the dynamical system $T : X \to X$ is said to be stable almost everywhere (a.e.) with respect to a finite measure $m \in \mathscr{M}(A^c)$ if

$$m\{x \in A^c : \omega(x) \not\subseteq A\} = 0 \tag{1}$$

where $\omega(x)$ denotes the set of ω-limit points for x and m is generally taken to be the Lebesgue measure. ■

These and other notions of stability were investigated with the aid of a certain stochastic operator, the Perron-Frobenius operator $\mathbb{P}$ corresponding to the dynamical system T. For a continuous mapping $T : X \to X$, it is given by

$$\mathbb{P}[\mu](A) = \mu(T^{-1}(A)), \tag{2}$$

where $\mu \in \mathscr{M}(X)$ and $A \in \mathscr{B}(X)$; cf., [17]. The Perron-Frobenius operator is useful because it defines a linear operator for a general nonlinear map. While, evolution of points is nonlinear and chaotic, the evolution of measures is linear and well-behaved.

In order to study stability of a given attractor set $A \subset X$ with respect to the initial conditions lying in the complement set, we considered the restriction of the Perron-Frobenius operator to measures supported on complement set $A^c \doteq X \setminus A$. In particular, the restriction $\mathbb{P}_1 : \mathscr{M}(A^c) \to \mathscr{M}(A^c)$ is shown to be a well-defined sub-Markov operator. With respect to this operator, a Lyapunov measure – that is particularly suited for stability analysis – is defined:

Definition 2 (Lyapunov measure): is any non-negative measure $\bar{\mu} \in \mathscr{M}(A^c)$ which is finite on $\mathscr{B}(X \setminus U(\varepsilon))$ and satisfies

$$\mathbb{P}_1\bar{\mu}(B) < \alpha\bar{\mu}(B), \tag{3}$$

TABLE I

CONDITIONS FOR RECURRENCE AND TRANSIENCE

	Linear (A)	Nonlinear (P_0, P_1)
Invariant set	$0 = A \cdot 0$	$\mu = \mu \cdot P_0$
Spectral condition	$\rho(A) < 1$	$\rho(P_1) < 1$
Series-expansion	$A^T \cdot P \cdot A - P = -Q$	$\bar{\mu} = m \cdot (I - P_1)^{-1}$
Linear inequalities	$--$	$\bar{\mu} \cdot P_1 < \bar{\mu}$

for every set $B \subset \mathscr{B}(X \setminus U(\varepsilon))$ and for every $\varepsilon > 0$ where $\bar{\mu}(B) > 0$ and $\alpha \leq 1$ is some positive constant.■

The connection between the Lyapunov measure and stability is given by the following Lyapunov theorem:

Theorem 3: Consider a continuous non-singular map $T : X \to X$ on a compact set $X \subset \mathbb{R}^m$ with an attractor set $A \subset V \subset X$. Suppose there exists a Lyapunov measure (Definition 2) with $\alpha < 1$, then

1) A is a.e. stable with respect to any finite measure m which is absolutely continuous with respect to Lyapunov measure $\bar{\mu}$.
2) A is a.e. stable with geometric decay (see [14] for definition) with respect to any measure m satisfying $m \leq \gamma\bar{\mu}$ for some constant $\gamma > 0$.

Proof: See [14]. ■

For cases where Lyapunov functions exist, Lyapunov measure is a dual to the Lyapunov function.

One of the advantages of using stochastic duality arguments is that they lead to efficient computational approaches for stability analysis. In particular, stability verification can be carried out by numerically obtaining finite-dimensional approximations of Lyapunov measures. The computational technique rests on set-oriented methods of Dellnitz and his coauthors [18], where one uses a finite partition of X to construct a Galerkin approximation of the P-F operator. These partitions can be chosen by taking quantization for states in X and the resulting P-F approximation arises as a Markov matrix. The approximation of the Lyapunov measure is computed using this Markov matrix (see Table I). For a stable attractor, such an approximation exists but the converse is not true. The existence of a Lyapunov measure for a Markov matrix typically only leads to a weaker notion of the stability, termed as coarse stability in [14]. For *typical* partitions, coarse stability means stability modulo dynamic behavior that is smaller than the size of cells within the partition. Table I gives several equivalent characterizations for verifying coarse stability using the Markov matrix and associated Lyapunov measures. We refer the reader to [14] for a discussion (including examples) on this.

III. Control-problem formulation

The stabilization problem is considered for a dynamical system with input

$$x_{n+1} = T(x_n, u_n), \tag{4}$$

where $x_n \in X$ and $u_n \in U$. To accommodate such maps, we discuss a straightforward generalization of the PF formalism to this case. Denoting $Y \doteq X \times U$ to be the product space, we assume that the map $T : Y \to X$. In general, this condition may be relaxed but the main ideas can be described more clearly with this assumption. As before, let $\mathscr{B}(Y)$ denote the Borel σ-algebra on Y and $\mathscr{M}(Y)$ the vector space of bounded real-valued measures on $\mathscr{B}(Y)$. The PF operator for T, denoted by $\mathbb{P}_T$, is given by

$$\mathbb{P}_T[\mu](A) = \int_Y \chi_A(Ty)d\mu(y), \tag{5}$$

where $\mu \in \mathscr{M}(Y)$, $A \in \mathscr{B}(X)$ and $\mathbb{P}_T[\mu] \in \mathscr{M}(X)$. Note $\mathbb{P}_T : \mathscr{M}(Y) \to \mathscr{M}(X)$.

The stabilization problem for (4) is to construct the control map

$$u_n = K(x_n), \tag{6}$$

such that the closed-loop system

$$x_{n+1} = T(x_n, K(x_n)), \tag{7}$$

has desired (global) stability properties for some attractor set $A \subset X$. The attractor set A may already be present for an open loop system or it may be the result of some locally stabilizing control.

Denoting $C(x_n) = (x_n, K(x_n))$, the stabilization problem is considered for the closed-loop:

$$X \quad \overset{C}{\to} \quad Y \overset{T}{\to} \quad X, \tag{8}$$

where $C : X \to Y$ is due to control, $T : Y \to X$ is the plant and both are assumed to be continuous non-singular maps on compact sets X, Y. Formally, the two maps are associated with the P-F operators:

$$\begin{aligned} &\mathbb{P}_C : \mathscr{M}(X) \to \mathscr{M}(Y), \\ &\mathbb{P}_T : \mathscr{M}(Y) \to \mathscr{M}(X). \end{aligned} \tag{9}$$

The following Lemma describes the P-F operator corresponding to the closed-loop equation (8).

Lemma 4: The P-F operator for the closed-loop $T \circ C : X \to X$ (see Eq. (8)) is given by $\mathbb{P}_T \cdot \mathbb{P}_C$.

Proof: Since $T \circ C : X \to X$, the corresponding P-F operator denoted by $\mathbb{P}_{T\circ C}$ is given by

$$\begin{aligned} \mathbb{P}_{T\circ C}[\mu](B) &= \int_X \chi_B(T\circ Cx)d\mu(x) \\ &= \int_Y \chi_B(Ty)d\mathbb{P}_C\mu(y) \\ &= \mathbb{P}_T[\mathbb{P}_C\mu](B). \end{aligned} \tag{10}$$

■

It has already been noted that for a nonlinear mapping, the P-F operator is a linear operator on the space of measures. The above Lemma shows that furthermore the nonlinear composition of two mappings lead to linear multiplication of the corresponding operators on the space of measures.

Using Lemma (8), the strategy will be to replace the deterministic control problem by its stochastic counterpart. Just as a Lyapunov measure was used for verifying stability in [14], it is used here for obtaining a stabilizing control. In particular, let $A \subset X$ be a given invariant set (of T) for which we seek a stabilizing control. The stabilization objective is achieved by co-designing 1) control $\mathbb{P}_C$ and 2) a Lyapunov measure μ such that

$$\mu\mathbb{P}_C \cdot \mathbb{P}_T < \mu, \tag{11}$$

where μ is a non-zero Lyapunov measure defined on the complement set $A^c = X \setminus A$. We note that in general, Y will depend upon the choice of control space U and more importantly, there are constraints on the map C. In particular, $C(x) = (x, K(x))$, where $K(x) \in U$. For $\mu \in \mathscr{M}(X)$, this is accomplished by apriori restricting the measure $\mathbb{P}_C[\mu] \in \mathscr{M}(Y)$ to be of the form

$$\mathbb{P}_C[\mu](D) = \int_D dq(a|x)d\mu(x), \tag{12}$$

where $q(a|x)$ is a conditional probability measure defined for each fixed $x \in X$ and $D \in \mathscr{B}(Y)$. As discussed in the following section, the control design problem will be to synthesize $q(a|x)$ together with Lyapunov measure μ (that satisfies (11)).

IV. Computational Approach

For the purposes of computations, the infinite-dimensional stochastic description of Eq. (11) is replaced by its finite-dimensional approximation. We assume a switched system formulation with $u \in \mathscr{U}_M$ where

$$\mathscr{U}_M = \{u_1, \dots, u_a, \dots, u_M\}, \tag{13}$$

a discrete set with finitely many (M) values. This set may be taken after quantization of the control input space.

We also assume a finite partition of X and denote it by $\mathscr{X}_L$, together with the associated measure space $\mathbb{R}^L$. The partition for the joint space Y, denoted by $\mathscr{Y} = \mathscr{X}_L \times \mathscr{U}_M$, has cardinality $M \cdot L$ and is identified with an associated vector space $\mathbb{R}^{ML}$. We use the notation P_C to denote the finite-dimensional counterpart (with respect to the discrete partition) of $\mathbb{P}_C$. We note that the entries of this matrix $P_C : \mathbb{R}^L \to \mathbb{R}^{ML}$ need to be designed to achieve the objective of stability. As before $P_T : \mathbb{R}^{ML} \to \mathbb{R}^L$ denotes the discrete counterpart of $\mathbb{P}_T$. Since $T : Y \to X$, so P_T is a Markov matrix.

Without loss of generality (by re-indexing perhaps), we assume that

$$\mathscr{X} \doteq \{D_1, \cdots, D_N\}, \tag{14}$$

is a sub-partition contained in the complement set A^c. We will synthesize a Lyapunov controller with respect to this partition. Before doing so, we will need to assume that the sub-partition $\{D_{N+1}, \dots, D_L\}$ is invariant under closed-loop. Note that by construction, $A \subset \cup_{j=N+1}^{L} D_j$. So, this property can be ensured by either constructing a fine enough partition in the neighborhood of an attractor set or if A is an unstable invariant set then by constructing a locally stabilizing controller. The Lyapunov-based design aims to construct a control and a Lyapunov measure that ensures global stability with respect to the complement set.

The control problem is then formulated as a co-design of 1) control Markov matrix $P_C : \mathbb{R}^N \to \mathbb{R}^{ML}$, and 2) an approximation of the Lyapunov measure $\underline{\mu} \in \mathbb{R}^N$ such that

$$\underline{\mu} P_C \cdot P_T[1:N] < \alpha \underline{\mu}, \tag{15}$$

where $\alpha \leq 1$. This is the discrete counterpart of Eq. (11).

This structure is too general because P_C can only correspond to control maps of the form $C(x) = (x, K(x))$. Below we introduce notation, counterpart of (12), to incorporate this. For each fixed value of control $u_a \in \mathscr{U}$, we denote $P^a : \mathbb{R}^L \to \mathbb{R}^L$ to be the Markov matrix for the map $T(\cdot, u = u_a)$. In particular, P^a are sub-matrices of P_T. Next, define

$$Q_{ia} = \mathrm{Prob}(u_n = u_a | x_n \in D_i) \text{ for } a \in [1, \ldots, M] \tag{16}$$

to be the probability of choosing the a^{th} control value conditioned on state being in cell D_j. Q is the discrete counterpart of conditional distribution $q(a|x)$ in Eq. (12). We note that Q_{ia} describes all of the non-zeros entries of P_C for control maps of the form $C(x) = (x, K(x))$. With this notation, Eq. (15) is written as

$$\sum_{a=1}^{M} \sum_{i=1}^{N} \mu_i Q_{ia} P^a_{ij} < \mu_j \text{ for } j = 1, \ldots, N. \tag{17}$$

The co-design problem thus involves designing a Markov matrix $Q : \mathbb{R}^N \to \mathbb{R}^M$ for control K together with the Lyapunov measure $\underline{\mu}$.

To solve this problem, we make a change of co-ordinate

$$R_{ia} \doteq \mu_i Q_{ia} \quad \text{for} \quad i \in [1, \ldots, N], \quad a \in [1, \ldots, M]. \tag{18}$$

By virtue of the fact that Q is a Markov matrix, we have

$$\mu_j = \sum_a R_{ja}. \tag{19}$$

The Lyapunov equation (15) for the closed-loop is

$$\sum_{i,a} R_{ia} P^a_{ij} < \alpha \sum_a R_{ja}, \quad \text{for} \quad j \in [1, \ldots, N]. \tag{20}$$

a linear inequality in the unknowns $\{R_{ia}\}$. Since the Lyapunov measure is positive and Q non-negative, this is augmented by constraints

$$R_{ia} \geq 0. \tag{21}$$

The equations (20)-(21) thus represent a system of linear inequalities in unknowns R_{ia}. A feasible solution for this can be obtained using linear programming. From any admissible solution to the linear program, the Lyapunov measure and control is easily obtained as

$$\begin{aligned} \mu_i &= \sum_a R_{ia}, & (22) \\ Q_{ia} &= \frac{R_{ia}}{\mu_i}. & (23) \end{aligned}$$

We note that the control Markov matrix Q is stochastic in general. In particular, a solution to Eq. (23) in general leads to $Q_{ia} \in [0,1]$. This is not surprising given our stochastic approach to the problem. This also has some advantage, namely due to expansion of the control space. Finally, a stabilizing solution exists as long as the feasibility set as defined by equations (20)-(21) is not empty.

A. Existence of a Stochastic Controller

The existence of a stochastic controller is linked to whether a solution exists to the set of inequalities given by

$$\begin{aligned} &\sum_{i,a} R_{ia} P^a_{ij} < \alpha \sum_a R_{ja}, \qquad j = 1, \ldots, N \\ &R_{ia} \geq 0, \qquad \forall i, a. \end{aligned} \tag{24}$$

Denoting $R \doteq [R^1, R^2, \ldots, R^M]'$, where $R^a \doteq [R_{1a}, R_{2a}, \ldots, R_{Na}]$, these inequalities may be succinctly expressed as

$$\begin{aligned} AR &> 0, \\ R &\geq 0, \end{aligned} \tag{25}$$

where $A = \begin{pmatrix} (\alpha I - P^1)' & \ldots & (\alpha I - P^M)' \end{pmatrix}$. The existence question for this linear system is easily resolved by using the Slater's theorem of alternative which we state below:

Theorem 5 ([19]): Let A be a given matrix. Then either $X := \{x : Ax > 0, x \geq 0\}$ is nonempty or $Y := \{y : A^T y^1 + y^2 = 0, y \geq 0\}$ is nonempty but never both.

Before stating and proving the necessary and sufficient conditions for the existence of a stochastic controller, we define e^Ω to be an *indicator vector* in $\mathbb{R}^m$. Here, $\Omega \subset \mathscr{X}$ and $e^\Omega_i = 1$ by definition if and only if $i \in \Omega$. We now state the existence result:

Theorem 6: Consider P^u to be a set of (in general) sub-Markov matrices and let

$$\beta = \min_u \lambda_{\max}(P^u), \tag{26}$$

where $\lambda_{\max}(\cdot)$ denotes the maximum (positive) eigenvalue. We have the following two cases regarding the existence of a solution to (24):

1) Suppose $\beta < 1$ then at least one solution exists for any $\alpha \in (\beta, 1]$.
2) Suppose $\beta = 1$ then a solution exists (in particular for $\alpha = 1$) if and only if

$$P^u e^\Omega = e^\Omega \quad \forall\, u = 1, \ldots, M \tag{27}$$

holds with only $\Omega = \phi$ the empty set. If solution to (24) exists for $\alpha = 1$ then it also exists for values of α sufficiently close but strictly smaller than 1.

Proof: Let $A \doteq \begin{pmatrix} (\alpha I - P^1)' & \ldots & (\alpha I - P^M)' \end{pmatrix}$, X denotes the set of feasible R defined by $X := \{R : R \geq 0, AR \geq 0\}$ and $Y := \{y : A^T y^1 + y^2 = 0, y > 0\}$. In order to prove the existence of a solution to X, we need to show that either X is non-empty or equivalently Y is empty. This follows from Slater's theorem of the alternative (see Theorem 5).

Suppose Y is non-empty then there exists an element of Y satisfying

$$\begin{aligned} y^2 &= -A^T y^1 \\ &= \begin{pmatrix} P^1 - \alpha I \\ \vdots \\ P^M - \alpha I \end{pmatrix} y^1 > \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, we have $P^a y_1 > \alpha y_1$ for all $a \in \{1,\ldots,m\}$. Then by corollary 8.1.29 from [20], we have $\alpha < \lambda_{max}(P^a)$ for $a \in \{1,\ldots,m\}$ or

$$\alpha < \min_a \lambda_{max}(P^a).$$

If α is set such that $\alpha \geq \min_a \lambda_{max}(P^a)$, then Y is empty, implying that X is nonempty and a stochastic controller may be designed. This concludes the proof for (1).

For part (2), assume $\beta = 1$. By Theorem 5, a solution exists if and only if there is no common $y \geq 0$ and not identically zero such that

$$P^u y = y \quad \forall\, u = 1,\ldots,M. \tag{28}$$

Denote $\Omega = \{i : y_i > 0\}$. Now, for a sub-Markov matrix $P^u y = y$ if and only if $P^u e^\Omega = e^\Omega$. The part (2) follows. Finally if a solution exists for $\alpha = 1$ then $AR < 0$ where the inequality is strict. Since the system is finite, this also implies $AR < -bR$ for sufficiently small positive b. Thus, a solution exists for all values of $\alpha \in [1-b, 1]$. ■

The part (1) of this theorem is expected. It merely states that if atleast one of the matrices P^{u_i} has largest eigenvalue strictly less than 1 then a stabilizing control always exists. This also follows by simply choosing the control to be open-loop ($u = u_i$ for all states). The part (2) of the theorem is more interesting. It states that the only way for a stochastic control not to exist is that there be a set $\Omega \subset \mathscr{X}$ that is left invariant by each of the matrices P^u. Intuitively, this statement makes sense. The interesting case thus arises where there are one or more invariant regions Ω^u for each choice of control (so $\beta = 1$) but one can still solve the co-design problem by using feedback. Equation (24) describes the set of feasible controls. We note that even if $\beta < 1$, the corresponding open-loop control (with $u = u_i$) may not be desirable due to control authority or optimality considerations. Equation (24) provides for a feasibility set of all stabilizing controllers for any choice of α. One may then obtain or choose a controller from this set that is better.

B. Constructing a deterministic controller

The linear programming formulation typically admits a stochastic controller. For deterministic control problems, this may not always be desirable. In many situations, it may be of interest to obtain deterministic control to the stabilization problem. This requirement imposes a constraint on the control Markov matrix Q to be deterministic.

Definition 7 (Deterministic Markov matrix [17]): A Markov or a sub-Markov matrix Q is **deterministic** if the individual entries are either 0 or 1. ■

Thus for the control to be deterministic, at most one entry of the i^{th} row of Q can be non-zero. Since the Lyapunov equation is given in terms of the co-ordinate R_{ia} (see Eq. (20)), we incorporate the constraint of deterministic control in terms of this co-ordinate.

This section presents one computational approach that leads to a mixed integer programming problem. For each unknown R_{ia}, we introduce a binary variable denoted by $y_{ia} \in \{0,1\}$. In order to obtain a deterministic control, we augment the system of linear inequalities in Eq. (20)-(21) by an additional set of linear inequalities

$$\begin{aligned} R_{ia} &\leq Z \cdot y_{ia}, \quad \text{for } i \in [1,\ldots,N], \\ &\qquad\qquad \text{and } a \in [1,\ldots,M], \end{aligned} \tag{29}$$

$$\sum_a y_{ia} = 1, \quad \text{for } i \in [1,\ldots,N] \tag{30}$$

where Z is a suitably large real constant. The equality constraint ensures that only one y_{ia} is non-zero for each i and the inequality constraint then ensures that R_{ia} is zero for all but one entry per i. Using Eq. (23), the control matrix is deterministic. In summary, equations (20)-(21) together with (29)-(30) and requirement that

$$y_{ia} \in \{0,1\} \tag{31}$$

defines the mixed-integer problem to obtain a deterministic control solution and a Lyapunov measure. In practice, a mixed integer problem is more complex and we plan to use LP relaxation based techniques [21].

V. EXAMPLE

Consider control of the 1-d cubic logistic map

$$x_{n+1} = T(x_n, u_n; \lambda) = \lambda x_n - x_n^3 + u_n, \tag{32}$$

with $\lambda = 2.3$, $X = [-1.6, 1.6]$ and a finite number of control values from

$$U = \{-0.15, -0.1, -0.05, 0, 0.05, 0.1, 0.15\}. \tag{33}$$

The value of λ is chosen to be at the *edge* where a sequence of period-doubling bifurcations lead to chaos. $\mathscr{X}$ is obtained by partitioning X into 101 equal-sized cells. The cell containing 0 is denoted as D_1. The control objective is to stabilize the unstable equilibrium at 0 by appropriately choosing $u_n = K(x_n)$ from U. Figure 1(a) depicts the invariant measure (in red) corresponding to the two symmetric attractors for the open-loop together with the invariant measure of the closed-loop (in blue). X_0 denotes the support of the latter and $X_1 = X \setminus X_0$. The control design problem considered here consists of co-designing a Lyapunov measure together with a deterministic control law $K(x)$ for points $x \in X_1$. In the following, we discuss the results for mixed-integer formulation (equations (20)-(21) and (29)-(31)) that was used to obtain a deterministic control.

Figure 1(b)-(d) summarizes the results of the control design with $\alpha = 1$. The part (b) depicts the control law $K(x)$ with respect to the partition where bars are used to reflect the fact that control takes a constant value on each cell. The part (c) gives the Lyapunov measure μ (in red) and μP (in blue) where P denotes the Markov approximation of the closed-loop dynamical system. The part (c) depicts the structure of the open and closed-loop Markov matrices (in red and blue respectively). The structure describes evolution of initial conditions under one iteration of the associated map. In the limit of taking infinite number of cells such that all points in X are distinguished, the structure will be identical to the graph of the map.

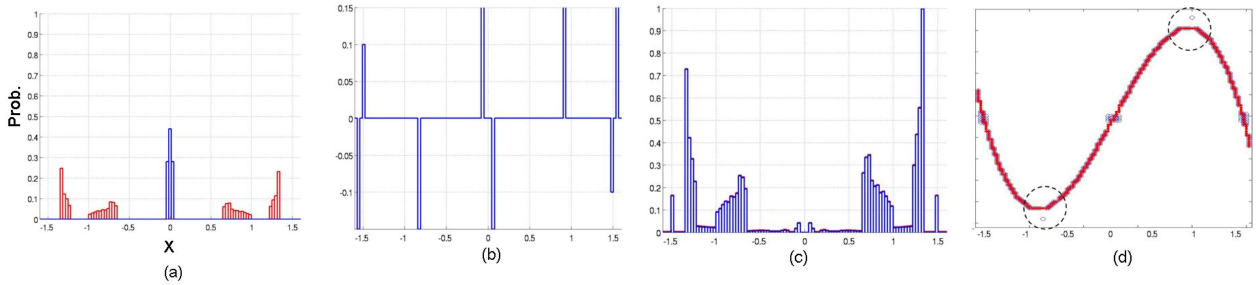

Fig. 1. (a) Invariant measure for the open-loop (in red) and the closed-loop (in blue), (b) control law, (c) Lyapunov measure $\underline{\mu}$, and (d) structure of the open-loop (in red) and closed-loop (in blue) transitions.

One remarkable fact about the control is that it utilizes the natural dynamics to aid the stabilization. The first observation is that the control is mainly inactive (see parts (b) and (d)). The part (c) shows that the control works by local stabilization and a relatively large control action in the region of the attractors for the open loop (indicated by the twin circles). The locally stabilizing nature of the control is reflected in the smaller slope for the closed-loop structure in the neighborhood of 0. The interesting observation is that the large control action does not work by pushing initial conditions in towards 0, rather just the opposite. In a sense, it is serving to both *break* the attractors of the logistic map and to utilize the natural dynamics that bring the initial conditions starting at the boundaries near 0 (note $T(\sqrt(\lambda),0;\lambda)=0$). This is also reflected in the "peaking" of the Lyapunov measure near the boundaries of X; cf. [14], [22] for interpretation of the peaks of the Lyapunov measure.

VI. Conclusion

In this paper, we developed an efficient computational framework for co-synthesis of a control-Lyapunov measure and the controller. We showed that the two notions of linearity and convexity implicit in LMI based linear control design are well captured using the stochastic formalism. The formalism of this paper thus allows one to generalize the intuition and results from linear systems to the general nonlinear dynamical systems.

The measure theoretic notions of stability also enable a generalization of the control design space. One, the solution of the linear inequalities in general lead to randomized control solutions for the stabilization problem (deterministic controllers can be also be obtained but require additional linear and integrality constraints). Two, the presence of unstable points in the complement set is typically useful for the stabilization problem. The notions of stability considered in this paper allow for such points. It even allows for dynamic behavior with small (than the quantization size) regions of attraction. The intuition is that such sets are either not important for the given scale or that large enough (size of quantization) noise makes them irrelevant. More generally, with the aid of a low-dimensional but chaotic example, we showed that the computational method provides a systematic framework to exploit nonlinear dynamics for control.

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