Lecture 6 A Primer on Random Processes

Recall that a *random variable, X,* is an *action.* If that action results in a single number, *x,* then *X* is a 1-D variable. If the action results in *n* numbers, $(x_1, x_2, \ldots, x_n)'$, then *X* is an *n*-D variable $(X_1, X_2, \ldots, X_n)'$.

Definition 1. A *continuous random process* is a collection of 'continuous-time' indexed random variables $\{X(t) \mid t \in (-\infty, \infty)\}$. A *discrete random process* is a collection of 'discrete-time' indexed random variables $\{X(k\Delta) \mid k \in \{-\infty, \infty\}\}$.

Example. A thermocouple is used to monitor the temperature of a lake near a nuclear power plant. A 24-hour measurement on 13 August, 2007 is shown below.

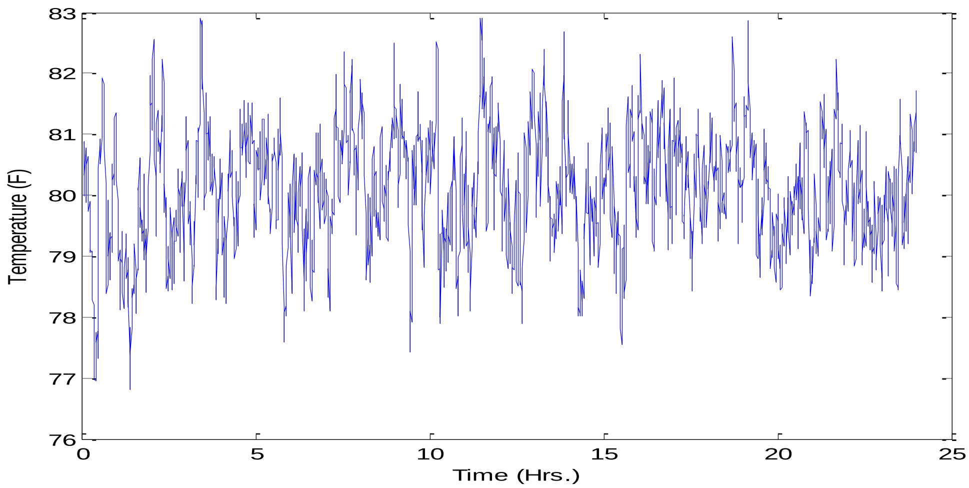


Figure 1. Sampled thermocouple output (°F) at lake location near power plant on 13 August 2007. Sampling interval Δ=0.01 Hrs.

What we have in Figure 1 is *data*. Specifically, we have 2400 numbers $\{w(k)\}_{k=1}^{24}$. We have a variety of actions that we can associate these numbers with:

(*i*) *X = The act of measuring temperature at any sample time.* This is a 1-D random variable, and we have 2400 measurements of it. From these, we can obtain information about *X*:

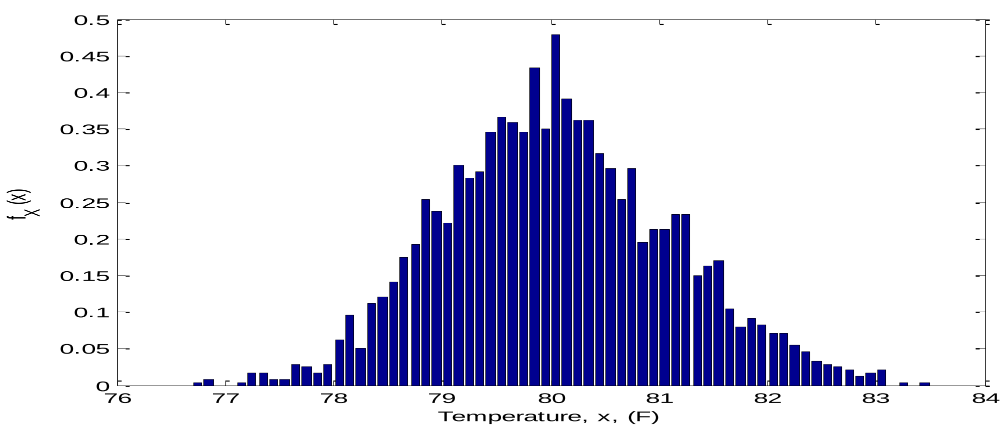


Figure 2. An estimate of $f_X(x)$ is shown in the plot. The estimates of μ_X and σ_X are 80.1 °F and 0.95 °F.

Based on Figure 2, we can claim that $X \sim N(80.1, 0.95^2)$.

(ii) *Y = The act of measuring temperature at the sample time just after that of X.* We now have a 2-D random variable (X,Y). We already know that $X \sim N(80.1, 0.95^2)$. If we ignore the relation of Y to X, then *marginally*, we have $Y \sim N(80.1, 0.95^2)$. The point of defining Y is that we desire to know if there is a relationship between successive temperature measurements. A scatter plot for (X, Y) based on the data in Figure 1 is shown below.

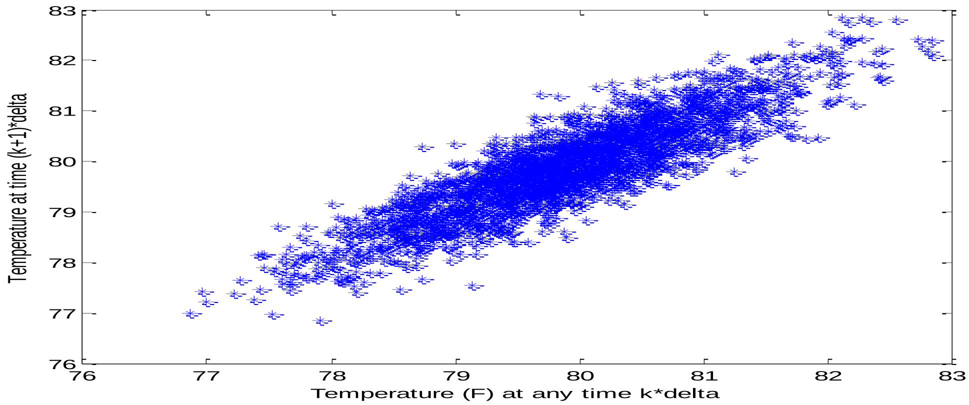
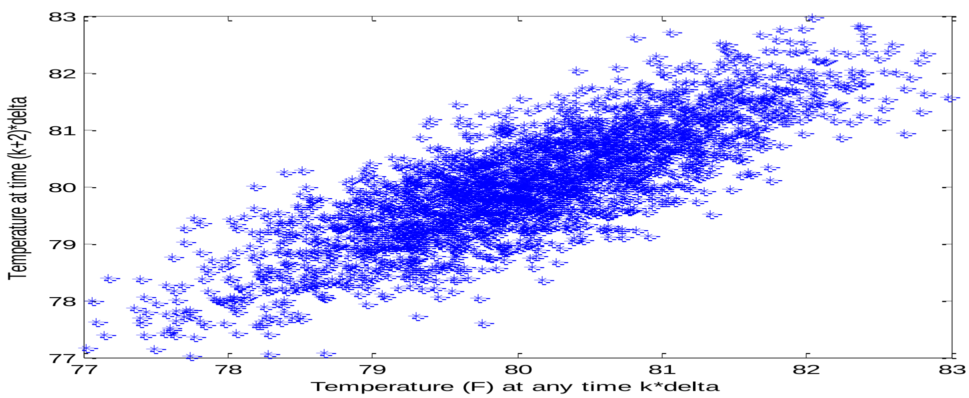


Figure 3. A scatter plot for (X, Y) based on the data in Figure 1.

There is clear evidence that X and Y have a positive correlation. We estimate that it as $\hat{\rho}_{XY} = 0.88$.

(iii) *Let Z = The act of measuring temperature at the sample time just after that of Y.* Here again, if we ignore the relation of Z to both X and Y, then $Z \sim N(80.1, 0.95^2)$. If we ignore the relation of Z to X, but focus on only its relation to Y, then $\rho_{YZ} = 0.88$. Hence, the point of bringing Z into the picture is that we desire to know about the relation between Z and X. A scatter plot for (X, Y) based on the data in Figure 1 is shown below.

Figure 3. A scatter plot for (X, Z) based on the data in Figure 1.



There is clear evidence that X and Z have a positive correlation. We estimate that it as $\hat{\rho}_{XZ} = 0.82$.

(iv) *Let* $\{W(k\Delta) \mid k \in \{-\infty, \infty\}\}$ *denote the act of measuring temperature over* <u>*any*</u> *infinite collection of sample times.* Then at any time $k\Delta$,the random variable $W(k\Delta)$, ignoring all other random variables is simply X in (i). More generally, for any time $k\Delta$,the random variables $\{W(k\Delta), W((k+1)\Delta), W((k+2)\Delta)\}$, ignoring all other random variables are exactly the random variables $\{X, Y, Z\}$. Were we to continue to define more and more random variables, based on how far apart in time any two measurements are, we would arrive at the random process $\{W(k\Delta) \mid k \in \{-\infty, \infty\}\}$. This way of defining random variables only in terms of how far apart they are from each other in time, with no concern for the absolute value of time, is tantamount to **assuming** the following properties of the temperature random process:

(P1) For any sample time $k\Delta$, $W(k\Delta) \sim N(80.1, 0.95^2)$

(P2) $Corr[W(k\Delta), W((k+n)\Delta)] = \rho_W(n\Delta)$, where $\rho_W(0) = 1, \quad \rho_W(1) = .88, \quad \rho_W(2) = .82$

Definition 2. A random process $\{X(k\Delta) \mid k \in \{-\infty, \infty\}\}$ is said to be ***wide sense stationary* (*wss*)** if the following two conditions hold:

(C1): For any sample time $k\Delta$, $E[X(k\Delta)] = \mu_X$ and (C2): $Corr[X(k\Delta), X((k+n)\Delta)] = \rho_X(n\Delta)$,

What we see is that:

*by defining random variables only in relation to time separation (ignoring absolute time) we have **forced** the random process* $\{W(k\Delta) \mid k \in \{-\infty, \infty\}\}$ *to be a wss process!!!*

Before we address the validity of this 'enforcement', let's now assume that the temperature process is, indeed, *wss*. We will now proceed to create a *model* for this process. This model will be based on the short-delay trend in the values of the autocorrelation function.

Specifically, from the above we have:

(i) $X(k\Delta) \sim N(80.1, 0.95^2) \quad \forall k$

(ii) $\hat{\rho}_{XY} = \hat{\rho}_X(\Delta) = 0.88$ and $\hat{\rho}_X(2\Delta) = 0.82$.

Definition 3. For a *wss* discrete-time random process, $\{X_k \overset{\Delta}{=} X(k\Delta)\}$, the ***autocorrelation function*** is defined as

$$R_X(m) \overset{\Delta}{=} E(X_k X_{k+m}). \tag{1}$$

In relation to the temperature random process, we can define the related process $\{Y_k \overset{\Delta}{=} Y(k\Delta)\}$ via:

$$X_k = \mu_X + Y_k. \tag{2}$$

From (2) it should be clear that $\{Y_k \overset{\Delta}{=} Y(k\Delta)\}$ is a *zero-mean wss* random process. Substituting (2) into (1) gives:

$$R_X(m) \overset{\Delta}{=} E(X_k X_{k+m}) = E[(\mu_X + Y_k)(\mu_X + Y_{k+m})] = \mu_X^2 + E(Y_k Y_{k+m}) = \mu_X^2 + R_Y(m). \tag{3}$$

From (3) we see that $R_X(m)$ is simply a shifted version of $R_Y(m)$. Hence, in order to arrive at a model for $\{X_k \overset{\Delta}{=} X(k\Delta)\}$ we will develop a model for $\{Y_k \overset{\Delta}{=} Y(k\Delta)\}$, and then simply add μ_X to Y_k to obtain a model for (2).

From the above, we have:

(i) $Y(k\Delta) \sim N(0, 0.95^2) \quad \forall\, k$. In other words: $\quad R_Y(0) = .95^2 = .9025 \cong 0.9$

(ii) $\hat{\rho}_Y(\Delta) = 0.88$ and $\hat{\rho}_X(2\Delta) = 0.82$.

In other words: $R_Y(1) = R_Y(0)\hat{\rho}_Y(1) = .9(.88) \cong .79$ and $R_Y(2) = R_Y(0)\hat{\rho}_Y(2) = .9(.82) \cong .74$

So, let's try the 'first order' linear prediction model: $\hat{Y}_k = \alpha Y_{k-1}$. Recall, that we obtain the value for the parameter α by solving: $E[(Y_k - \hat{Y}_k)Y_{k-1}] = 0$. Specifically,

$E[(Y_k - \hat{Y}_k)Y_{k-1}] = 0 = E(Y_k Y_{k-1} - \alpha Y_{k-1}^2) = R_Y(1) - \alpha\, R_Y(0)$. Hence,

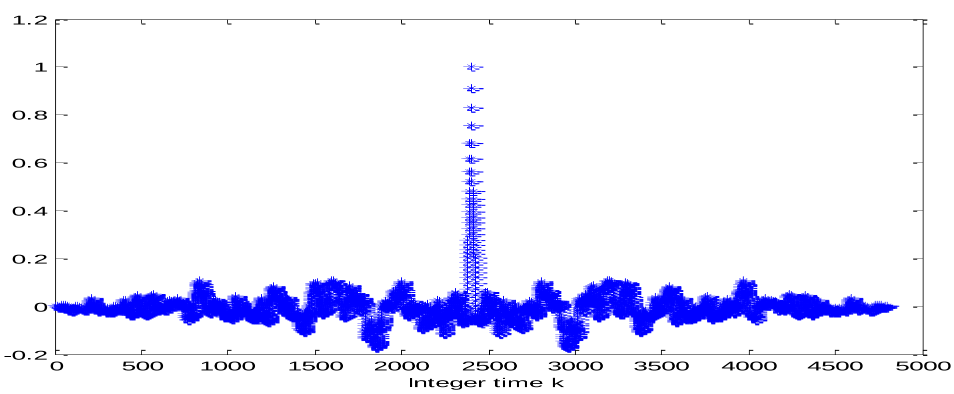
$$\alpha = R_Y(1) / R_Y(0)\,. \tag{3}$$

Furthermore, by defining $\quad U_k \overset{\Delta}{=} Y_k - \hat{Y}_k = Y_k - \alpha Y_{k-1}$

we arrive at the model: $$Y_k = \alpha Y_{k-1} + U_k\,. \tag{4}$$

IF in the model (4), the error process $\{U_k\}$ is a *white noise* process (i.e. the random variables are *iid*) then the process (4) is called an AR(1) process (i.e. an Auto-Regression process of order 1).

The *Matlab* command xcorr.m is designed to compute $Corr[X(k\Delta), X((k+n)\Delta)] = \rho_X(n\Delta)$. However, it is disappointing that, at this point in time, the above command must be augmented by a host of code to compute $\rho_X(n\Delta)$. One must be careful to first subtract the data mean, prior to using the above command. Upon doing this, for the de-meaned data in Figure 1, we obtain the figure below.



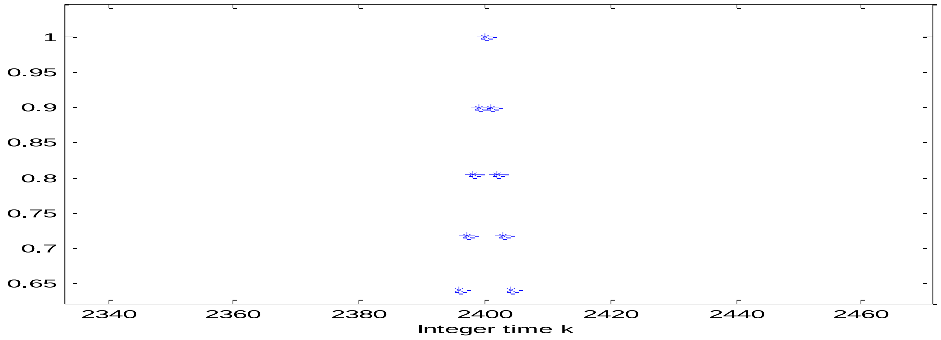


Figure 5. Computation of $\rho_W(n\Delta)$ obtained using the command xcorr(dw,'coef') where dw is the de-meaned data in Figure 1.

Notice that $\rho_W(0) = 1.0$ is placed in the center of the plot. Hence the center time index 2401 is really lag n=0. The plot is symmetric about this point since $\rho_W(n\Delta) = \rho_W(-n\Delta)$. In the lower (zoomed) plot in Figure 5, we observe the numerical values that we obtained above; namely, $\rho_W(\Delta) = \rho_{XY} = 0.88$ and $\rho_W(2\Delta) = \rho_{XY} = 0.82$.

Below is the *Matlab* code that was actually used to generate the time series in Figure 1.

```
%PROGRAM NAMES: temp.m
%lake temperature near nuclear plant for 24-hr period
t=0:.01:24-0.01;
nt = length(t);    *** simulate a length-n portion of the random process w(k) ***
dw=zeros(1,nt);
a = .9;
u=.19^.5*randn(1,nt);  Note that this loop is exactly the model: Y = aX + U
dw(1)=randn(1,1);      performed in a recursive fashion.
for k=2:nt
dw(k)=a*dw(k-1)+ u(k);
end
w=80+dw;               Note that when simulating a random process with a non-zero
figure(1)              mean, the mean is added AFTER the zero-mean process is generated.
plot(t,w)
xlabel('Time (Hrs.)')
ylabel('Temperature (F)')
pause
% Let X = act of measuring Temp at any time
mx = mean(w);
sigx=std(w);
bvec = 76.05:.1:83.5;
h = hist(w,bvec);
fx = (0.1*nt)^-1 *h;
figure(2)
bar(bvec,fx)
xlabel('Temperature, x, (F)')
ylabel('f_X (x)')
pause
% Let Y = act of measuring the next Temp after X
xy = [w(1:nt-1)' , w(2:nt)'];
figure(3)
plot(xy(:,1),xy(:,2),'*')
xlabel('Temperature (F) at any time k*delta')
ylabel('Temperature at time (k+1)*delta')
Rxy = corrcoef(xy);
rxy = Rxy(1,2);
pause
% Let Z = act of measuring the next temp after Y
```

```
xz = [w(1:nt-2)' , w(3:nt)'];
figure(4)
plot(xz(:,1),xz(:,2),'*')
xlabel('Temperature (F) at any time k*delta')
ylabel('Temperature at time (k+2)*delta')
Rxz = corrcoef(xz);
rxz = Rxz(1,2);
pause
% Compute the de-meaned autocorreltation function
dw = w - mean(w);
rdw = xcorr(dw,'coef');
figure(5)
plot(rdw,'*')
xlabel('Integer time k')
```