An Investigation of Rate Gyro Noise Properties

1. The Noise Associated with Rate Gyro Angular Velocity and Position

Angular Velocity-

Typically, the noise associated with a rate gyro angular velocity, $\dot{\theta}(t) = \omega(t)$ is assumed to be ***white noise***. In other words, it is a random process that has a flat *power spectral density* (*psd*). For example, suppose that for a chosen sampling interval, Δ seconds, the rate gyro noise has a rated standard deviation $\sigma_\omega = \sigma_o / \sqrt{\Delta} \;\; {}^o/\sec$.

Question: What are the units of σ_o? *Answer*: $\sigma_o = \sigma_\omega \sqrt{\Delta} \quad [({}^o/\sec) \times \sqrt{\sec}] \;\; = \;\; [{}^o/\sqrt{\sec}]$

Recall that the sampling frequency is $f_{samp} = 1/\Delta \; Hz$ Hence, the *Nyquist frequency* is $f_{Nyquist} = f_{samp}/2 = 1/2\Delta \; Hz$ Furthermore, suppose that it is assumed to have a flat *psd* over the analysis frequency range $[-f_{Nyquist}, f_{Nyquist}]Hz.$ Since the noise 'power' is defined as its variance, σ_ω^2, it follows from the conservation of energy that its *psd* is

$$S_\omega(f) = \sigma_o^2/(\Delta f_{samp}) = \sigma_o^2 \left[\frac{({}^o/\sec)^2}{Hz} \right]. \tag{1a}$$

It is important to note that the *psd* (1a) does not depend on Δ. For a given Δ, the power (i.e. variance) of the sampled signal $\omega(k\Delta)$ is, as we know, $\sigma_\omega^2(\Delta) = \sigma_o^2/\Delta$. Hence, as $\Delta \to 0$, $\omega(k\Delta \cong t) \to \omega(t)$, which is a *continuous-time* white noise process. And so we can readily conclude that such a process has *infinite* variance! Consequently, the parameter σ_o^2 is <u>not</u> variance; rather, it is a variance *intensity* parameter. Alternatively, it is the value of the *psd* (1a).

To obtain the value of (1a) in *decibels* (*dB*), we use

$$S_\omega(f)_{dB} = 10\log_{10} S_\omega(f) \; dB. \tag{1b}$$

Unfortunately, most rate gyros do not specify the *psd*. Instead, they specify that *amplitude spectral density* (*asd*), which is the square root of the *psd*. Thus, in relation to (1a), we have:

$$A_\omega(f) = \sqrt{S_\omega(f)} = \sigma_o \left[\frac{{}^o/\sec}{\sqrt{Hz}} \right]. \tag{1c}$$

Notice that the *dB* value of (1c) is still (1b), since when converting amplitude to *dB* one uses $20\log_{10}$.

Angular Position- Clearly, neglecting initial angular position, we have:

$$\theta(t) = \int_0^t \omega(\tau) d\tau. \tag{2a}$$

And so, (2a) becomes the approximation:

$$\theta(k\Delta) = \sum_{j=1}^{k} \omega(j\Delta)\Delta. \tag{2b}$$

Now, the random variables $\{\omega(j\Delta)\}_{j=1}^{k}$ each have variance $\sigma_\omega^2 = \sigma_o^2 / \Delta$. Moreover, the white noise assumption implies that they are mutually uncorrelated. Hence, the variance of (2b) is:

$$Var[\theta(k\Delta)] = Var\left[\sum_{j=1}^{k}\omega(j\Delta)\,\Delta\right] = \Delta^2 Var\left[\sum_{j=1}^{k}\omega(j\Delta)\right] = \Delta^2\sum_{j=1}^{k}Var[\omega(j\Delta)] = (k\Delta^2)(\sigma_o^2/\Delta)\,. \tag{3}$$

If we let $t_k \overset{\Delta}{=} k\Delta$, then (3a) becomes

$$Var[\theta(t_k)] = t_k\sigma_o^2\,. \tag{3b}$$

Equivalently,

$$Std[\theta(t_k)] \overset{\Delta}{=} \sigma_\theta(t_k) = \sigma_o\sqrt{t_k}\ \deg\,. \tag{3c}$$

Notice that as the time t_k increases, so does the uncertainty of the angular position $\theta(t_k)$ due to the rate gyro sensor noise. The random process $\{\theta(t_k)\}_{k=1}^{\infty}$ with this behavior is called an *angular random walk* (*ARW*).

Example 1. Suppose that a given rate gyro has a specified *ARW* ***uncertainty rate*** $r_{ARW} = .003^o/\sqrt{hr}$. Furthermore, it is claimed that after 6 minutes the uncertainty in angular position is 0.001°.

Question 1: How does the uncertainty rate relate to (3c)?

Answer: Recall from the above Q/A that $\sigma_o = \sigma_\omega\sqrt{\Delta}\quad [^o/\sqrt{\sec}]$. Hence, $r_{ARW} = \sigma_o$. In words, the *ARW* uncertainty rate is the standard deviation *intensity* of the *continuous-time* random walk $\theta(t)$. Equivalently, it is the *asd* (1c).

Question 2: For a sampling frequency $f_{samp} = 50Hz$, what is the numerical value of the standard deviation of the gyro rate, σ_ω?

Answer : $$r_{ARW} = .003^o/\sqrt{hr}\times\sqrt{\frac{1\,hr}{3600\sec}} = .5(10^{-4})\ ^o/\sqrt{\sec}\,.$$

Hence, $$\sigma_\omega = \sigma_o/\sqrt{\Delta} = \frac{.5(10^{-4})}{\sqrt{.02}} = 3.54(10^{-4})\ ^o/\sec\,.$$

Question 3: What is the relationship between $Std[\theta(t_k)] \overset{\Delta}{=} \sigma_\theta(t_k) = \sigma_o\sqrt{t_k}\ \deg$ and $\sigma_\omega = \sigma_o/\sqrt{\Delta}\ )\ ^o/\sec$.

Answer: $$\sigma_\theta(t_k) = \sigma_o\sqrt{t_k} = (\sigma_\omega\sqrt{\Delta}\,)\sqrt{k\Delta} = \sigma_\omega\Delta\sqrt{k}\,. \tag{4}$$

Question 4: Why is the relation (4) important if one desires to verify the manufacturer's claimed value for r_{ARW}?

Answer: It is important because we only have the sampled the rate signal, $\omega(k\Delta)$ at a specified sampling frequency, $f_{samp} = 1/\Delta$. Hence, we can only estimate σ_ω. We can never estimate σ_o directly. Even so, having an estimate $\hat{\sigma}_\omega$ allows us to estimate σ_o using $\hat{\sigma}_o = \hat{\sigma}_\omega \sqrt{\Delta}$.

Question 5: Some manufacturers specify r_{ARW} in units $^o / hr / \sqrt{Hz}$. Show the relation between $r_{ARW}^{(1)}$ $[^o / \sqrt{hr}]$ and $r_{ARW}^{(2)}$ $[^o / hr / \sqrt{Hz}]$.

Answer: Clearly, $r_{ARW}^2 = \sigma_o^2$ is the value of the *psd*, $S_\omega(f)$, given by (1a), with dimensions $\left[\frac{(angle/time)^2}{1/time}\right]$. Hence, $r_{AR} = \sigma_o$ has dimensions $\left[\frac{(angle/time)}{1/\sqrt{time}}\right]$. But we also know that this <u>same</u> $r_{AR} = \sigma_o$ has dimensions $\left[\frac{angle}{\sqrt{time}}\right]$. And so, the numerical value of $r_{AR} = \sigma_o$ can be given either of these sets of units, as long as the *time* units are one and the same!

Comment on the excerpt from

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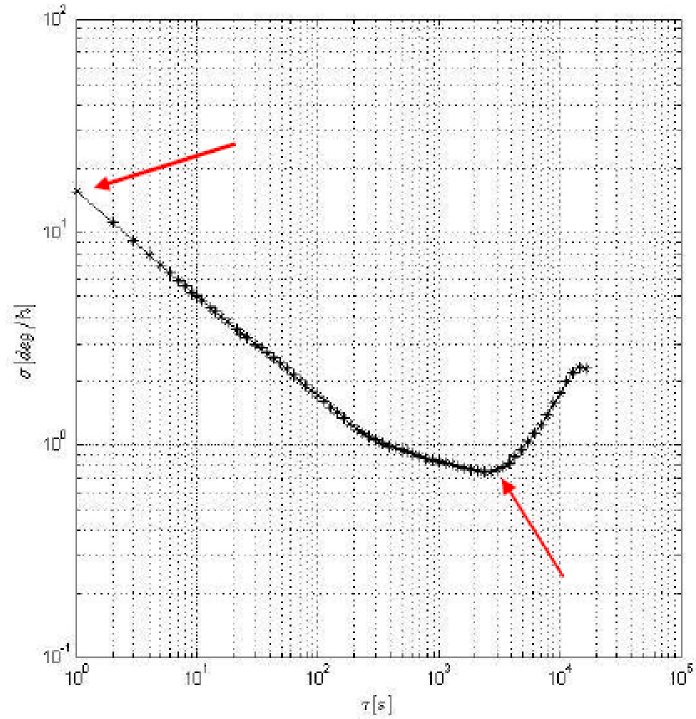
in relation to the above development.

Gyro Random Walk: This value, given in deg/sqrt(hr), shows the noise of the used gyro. The higher the noise the more noise is measured on the angular rates and on the angles. Some manufacturers also specify it as the noise density in deg/h/sqrt(Hz). Both values are equivalent - if the second value is divided by 60, you get it in deg/sqrt(hr). An angular random walk of 0.003 deg/sqrt(hr) indicates, that the angular error (incertainty) due to random walk is e.g. 0.001 deg after 6 minutes (unaided) or 0.0004 deg after 1 minute (all values one sigma). The angular random walk is very important for the accuracy of north seeking, because if the random walk decreases times 2 then the needed duration for north seeking decreases by times four (if the resolution of the gyro is high enough).



The plot of the Allan Variance shows the square-root ARW of a MEMS gyro graphically (take the value at 1 sec and divide it by sixty to obtain the ARW in [deg/sqrt(hr)]).

At 1 sec the value of the square-root of the AllanVariance is 15 deg/hr. This leads to a value of the Angular Random Walk (ARW) of 15/60 deg/sqrt(hr) = 0.25 deg/sqrt(hr) = 0.0042 deg/s/sqrt(Hz) = 15 deg/hr/sqrt(Hz) [white gyro noise assumed]. The bias stability (minimum point of the graph) is 0.8 deg/hr at a correlation time of 3'000 seconds. So it is really quite a good MEMS gyro which we have in use.

<u>Comment</u>: It is stated that the two values that manufacturers give are equivalent, but not equal. To see what's going on here, let's follow their suggested procedure:

Suppose that $r_{ARW}^2 = \sigma_o^2$ given as $\left[\dfrac{(^o)^2}{hr}\right]$. Since from (4), we have $\sigma_\theta^2(t) = \sigma_o^2 t$, then the time units associated with $\theta(t)$ must be the same as those associated with $r_{ARW}^2 = \sigma_o^2$ (i.e. hours here) for this variance to make any sense. Similarly, the *psd* frequency units must be cycles/hour. Otherwise the area under the *psd* will be dimensionally inconsistent.

This begs the question: Why would manufacturers present $r_{ARW}^2 = \sigma_o^2$ in units of $\left[\dfrac{(^o/hr)^2}{Hz}\right]$? One reason might be that it has simply become a convention.

In any case, if one is given $r_{ARW}^2 = \sigma_o^2$ in units $\left[\frac{(^o/hr)^2}{Hz}\right]$, then it is necessary to convert Hz to cycles/hour. Specifically:

$$\left[\frac{(^o/hr)^2}{Hz}\right]\times\left[\frac{1\,hr^2}{3600\sec^2}\right]=\frac{1}{60^2}\left[\frac{(^o/hr)^2}{cycles/hr}\right]=\frac{1}{60^2}\left[\frac{(^o)^2/hr}{cycle}\right].$$

Equivalently,

$$\left[\frac{^o/hr}{\sqrt{Hz}}\right]=\frac{1}{60}\left[\frac{^o/\sqrt{hr}}{\sqrt{cycle}}\right]=\frac{1}{60}\left[^o/\sqrt{hr}\right].$$

Conclusion: If you have a *psd* estimate for $\omega(k\Delta)$ where the units of Δ are seconds, then you need to convert those units to hours if you want the area under the curve to make sense. □

2. The Color of the Rate Noise

The above analysis assumed that the rate noise was white over the analysis bandwidth. Consider, however, the noise *psd* shown below.

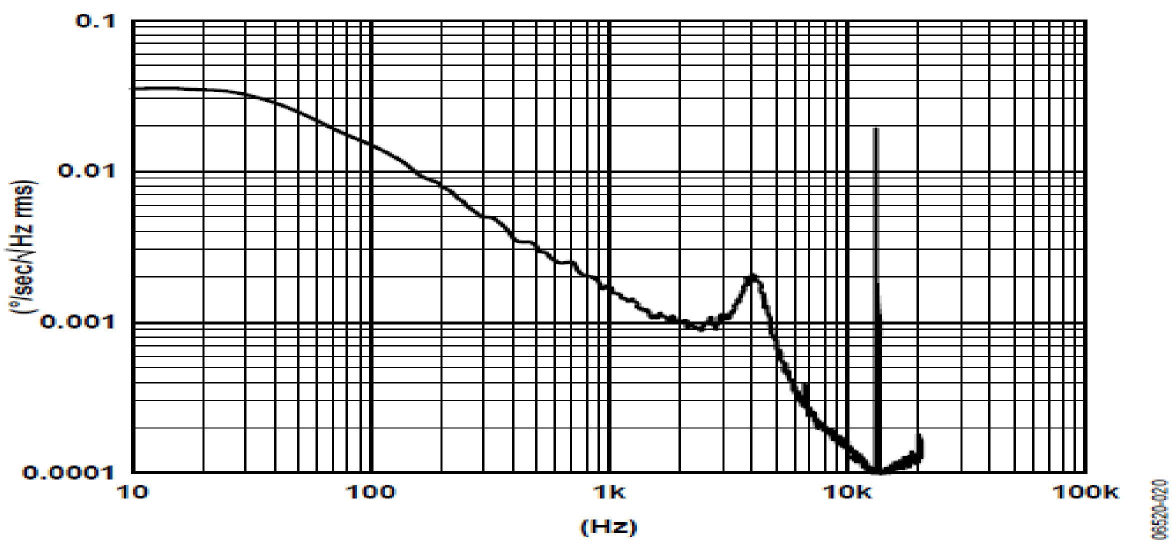


Figure 21. Typical Noise Spectral Density (Bandwidth = 40 Hz)

Clearly, when using a sufficiently high sampling rate, the noise is not at all white; but is highly ‘colored’. The above *psd* was obtained using $f_{samp} = 100kHz$.

Analysis of the Results Presented in:

mems *gyroscope* performance comparison using *allan variance*

[This site is apparently no longer active. ☹]

www.feec.vutbr.cz/EEICT/2011/sbornik/03.../03.../14-xvagne04.pdf

by M Vagner - Cited by 1 - Related articles

presented. Keywords: MEMS, gyroscope, *Allan variance*, stability, bias, random walk **...** The *rate gyroscope* output is disturbed by two main groups of errors.

2 ALLAN VARIANCE

As mentioned, AV is a method of analysis in a time domain. It describes variance of a signal as a function of averaging time. Frequently, the Allan variance term is also used to refer to its square root. We can also often see the term cluster analysis, which expresses the principle of operation. In the IEEE 952-1997 [1] standard, the Allan variance $\sigma^2_\Omega(\tau)$ is described for an angle velocity Ω as

follows:

$$\theta(t) = \int^t \Omega(t')dt' \tag{1}$$

$$\overline{\Omega}_k(\tau) = \frac{\theta(t_k+\tau)-\theta(t_k)}{\tau} \tag{2}$$

$$\sigma^2_\Omega(\tau) = \frac{1}{2}\left\langle \left(\overline{\Omega}_{k+1}-\overline{\Omega}_k\right)^2 \right\rangle , \tag{3}$$

where τ is an averaging time and operator $\langle ... \rangle$ is defined as an infinite time average [2]:

$$\langle f(t) \rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt \tag{4}$$

relation between power spectral density and Allan variance is expressed as [1]:

$$\sigma^2_\Omega(\tau) = 4\int_0^\infty S_\Omega(f)\frac{\sin^4(\pi f \tau)}{(\pi f\tau)^2}df \tag{6}$$

COMMENT: Since (2) is a r.v. it follows that for any T, (3) is also a r.v. Even so, by appealing to the psd and (3) it is assumed that as T goes to infinity, (3) is not a random variable. This is, in a word, sloppy. For one thing, a sequence of random variables, such as (3) as a function of k, can converge in different ways (e.g. almost surely, in probability, or in mean square senses). Also, it should be remembered that in (3) and (6) tau is a fixed value. Hence, the question arises:

QUESTION: Given a sequence of r.v.s $\{\overline{\Omega}_k(\tau)\}_{k=1}^{\infty}$, how is it possible that there exists a proper integral such as (4)?

It is clear that this sequence depends on the chosen sampling interval Δ. So is the above presuming implicitly that Δ is sufficiently small such that $\{\overline{\Omega}_k(\tau)\}_{k=1}^{\infty} \cong \{\overline{\Omega}_t(\tau)\}_{t\in[0,\infty)}$? But if this is presumed, then from (2) we have

$\bar{\Omega}_t(\tau) \cong [\theta(t+\tau) - \theta(t)]/\tau$. This can be viewed simply as $\bar{\Omega}_t(\tau) \cong \Omega(t)$ for small τ. For large τ I'm not sure what it is. I will be the first to confess that I am not well-versed in the Vann Allen variance (see below). However, I do have a modest understanding of random processes. And to me, the above is cryptic, at best. So, I can only imagine how cryptic it might be to a student. ☹ Yuck!

2.1.1 ANGLE RANDOM WALK (ARW)

ARW is high frequency noise and it can be observed as the short-term variation in the output. After performing an integration, it causes random error in angle with distribution, which is proportional to the square root of the elapsed time. As described at [1] ARW appears in PSD as:

$$S_\Omega(f) = N^2 \tag{8}$$

Substituting (8) into the equation (6) we get:

$$\sigma^2_{arw}(\tau) = \frac{N^2}{\tau} \tag{9}$$

The equation (9) shows that the slope of $\sigma(\tau)$ log-log plot is -0.5 and the coefficient N can be obtained from the plot at $\tau = 1$ (fig. 1).

We will now investigate the above results. To this end, we begin with equations (1-3):

Definition 1. The **Allan Variance** is defined as:

$$\sigma^2_\Omega(\tau) \quad \stackrel{\Delta}{=} \quad \frac{1}{2} E\left\{ \left[\bar{\Omega}_{k+1}(\tau) - \bar{\Omega}_k(\tau) \right]^2 \right\}$$

where $\bar{\Omega}_k(\tau) \quad \stackrel{\Delta}{=} \quad \dfrac{\theta(t_k+\tau) - \theta(t_k)}{\tau}$ (which makes no sense if Δ is sufficiently small)

$$\theta(t) \quad \stackrel{\Delta}{=} \quad \int_{\xi=0}^{t} \Omega(\xi)\, d\xi$$

and where $\Omega(t)$ is a continuous-time random walk with rate parameter σ_o.

Let $\Delta = t_{k+1} - t_k$ denote the sampling interval. Then

$$\bar{\Omega}_{k+1}(\tau) - \bar{\Omega}_k(\tau) = \frac{1}{\tau}\left[\int_{\xi=t_k+\Delta}^{t_k+\Delta+\tau} \Omega(\xi)\, d\xi - \int_{\xi=t_k}^{t_k+\tau} \Omega(\xi)\, d\xi \right]. \tag{1}$$

***Case* 1 ($\tau \le \Delta$):** In this case, the intervals $[t_k\ ,\ t_k+\tau]$ and $[t_k+\Delta\ ,\ t_k+\Delta+\tau]$ are *disjoint* intervals. Hence,

$$Var\left[\bar{\Omega}_{k+1}(\tau) - \bar{\Omega}_k(\tau)\right] = \frac{1}{\tau^2}\left\{ Var[\bar{\Omega}_{k+1}(\tau)] \quad + \quad Var[\bar{\Omega}_k(\tau)] \right\} = \frac{\sigma_o^2}{\tau^2}\{\tau \quad + \quad \tau\} = \frac{2\sigma_o^2}{\tau}. \tag{2}$$

***Case* 2 ($\tau > \Delta$):**

$$\int_{\xi=t_k+\Delta}^{t_k+\Delta+\tau}\Omega(\xi)\,d\xi-\int_{\xi=t_k}^{t_k+\tau}\Omega(\xi)\,d\xi \;=\; \int_{\xi=t_k+\Delta}^{t_k+\Delta+\tau}\Omega(\xi)\,d\xi-\int_{\xi=t_k+\Delta}^{t_k+\tau}\Omega(\xi)\,d\xi-\int_{\xi=t_k}^{t_k+\Delta}\Omega(\xi)\,d\xi \;=\; \int_{\xi=t_k+\tau}^{t_k+\Delta+\tau}\Omega(\xi)\,d\xi-\int_{\xi=t_k}^{t_k+\Delta}\Omega(\xi)\,d\xi\,. \qquad (3)$$

In this case, the intervals $[t_k\ , t_k+\Delta]$ and $[t_k+\tau\ , t_k+\Delta+\tau]$ are *disjoint* intervals. Hence,

$$Var\left[\overline{\Omega}_{k+1}(\tau)-\overline{\Omega}_k(\tau)\right]=\frac{1}{\tau^2}\left\{Var[\overline{\Omega}_{k+1}(\tau)] \;+\; Var[\overline{\Omega}_k(\tau)]\right\}=\frac{\sigma_o^2}{\tau^2}\left\{\Delta \;+\; \Delta\right\}=\frac{2\sigma_o^2\Delta}{\tau^2}\,. \qquad (4)$$

From (2) and (4) we arrive at:

$$\sigma_\Omega^2(\tau)=\begin{cases}\dfrac{\sigma_o^2}{\tau} & for\ \tau\le\Delta\\[2mm] \dfrac{\sigma_o^2\Delta}{\tau^2} & for\ \tau>\Delta\end{cases}$$

Equation (4) is a true variance. On the contrary, the use of $\langle f(t)\rangle=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}f(t)dt$ is an attempt to use ergodicity to claim that the this true variance can be obtained by time-averaging. It would be an interesting exercise to try to prove that this true variance really is $\sigma_\Omega^2(\tau)=4\int_0^\infty S_\Omega(f)\frac{\sin^4(\pi f\tau)}{(\pi f\tau)^2}df$. However, it is an exercise that I have no interest in pursuing at this time; given my frustrated state of mind with the above presentation of the topic. Let me only point out that the true variance is clearly a function of Δ; whereas Δ is nowhare to be found in this integral.