# Lecture 17 The Integral of a *wss* Process

Let $X(t)$ be a wide sense stationary (*wss*) zero-mean process with $E[X(t)X(t+\tau)] = R_X(\tau)$. Define the process

$$Y(t) = \int_0^t X(\upsilon)d\upsilon . \tag{1}$$

Clearly, the mean of (1) is zero. Furthermore,

$$R_Y(t,t+\tau) \stackrel{\Delta}{=} E[Y(t)Y(t+\tau)] = E\left[\int_{\upsilon=0}^{t} X(\upsilon)d\upsilon \int_{\eta=0}^{t+\tau} X(\eta)d\eta\right] = \int_{\upsilon=0}^{t}\int_{\eta=0}^{t+\tau} R_X(\eta-\upsilon)d\eta\, d\upsilon . \tag{2}$$

For any fixed $\upsilon$ let $\rho = \eta - \upsilon$. Then $d\rho = d\eta$, and (2) becomes

$$R_Y(t,t+\tau) = \int_{\upsilon=0}^{t}\int_{\rho=-\upsilon}^{t+\tau-\upsilon} R_X(\rho)d\rho\, d\upsilon \quad = \quad \int_{\upsilon=0}^{t}\left(\int_{\rho=-\upsilon}^{0} R_X(\rho)d\rho \quad + \quad \int_{\rho=0}^{t+\tau-\upsilon} R_X(\rho)d\rho\right)d\upsilon . \tag{3}$$

Since $R_X(\rho) = R_X(-\rho)$, we can write (3) as

$$R_Y(t,t+\tau) = \int_{\upsilon=0}^{t}\left(\int_{\rho=-\upsilon}^{0} R_X(-\rho)d\rho \quad + \quad \int_{\rho=0}^{t+\tau-\upsilon} R_X(\rho)d\rho\right)d\upsilon . \tag{4}$$

Let $\eta = -\rho$. Then $-d\eta = d\rho$, and (4) becomes

$$R_Y(t,t+\tau) = \int_{\upsilon=0}^{t}\left(\int_{\eta=0}^{\upsilon} R_X(\eta)d\eta \quad + \quad \int_{\rho=0}^{t+\tau-\upsilon} R_X(\rho)d\rho\right)d\upsilon . \tag{5}$$

Now define

$$\breve{R}_X(t) \quad \stackrel{\Delta}{=} \quad \int_{\eta=0}^{t} R_X(\eta)d\eta . \tag{6}$$

Then (5) becomes

$$R_Y(t,t+\tau) = \int_{\upsilon=0}^{t}\left(\breve{R}_X(\upsilon) \quad + \quad \breve{R}_X(t+\tau-\upsilon)\right)d\upsilon . \tag{7}$$

Next, define

$$\breve{\breve{R}}_X(t) \overset{\Delta}{=} \int_{\eta=0}^{t} \breve{R}_X(\eta) d\eta \,. \tag{8}$$

Then (7) becomes

$$R_Y(t, t+\tau) = \breve{\breve{R}}_X(t) + \int_{\upsilon=0}^{t} \breve{R}_X(t+\tau-\upsilon)\, d\upsilon \,. \tag{9}$$

Let $\eta = t + \tau - \upsilon$. Then $d\eta = -d\upsilon$, and (9) becomes

$$R_Y(t, t+\tau) = \breve{\breve{R}}_X(t) + \int_{\eta=\tau}^{t+\tau} \breve{R}_X(\eta)\, d\eta \;. \tag{10}$$

Hence,

$$R_Y(t, t+\tau) = \breve{\breve{R}}_X(t) + \breve{\breve{R}}_X(t+\tau) - \breve{\breve{R}}_X(\tau) \tag{11}$$

From (11) we see that (1) is not *wss*.

**Example 1.** Let $X(t)$ be a *Gauss-Markov* (*GM*) process with $R_X(\tau) = \sigma_X^2 e^{-\beta|\tau|}$. Then (6) is:

$$\breve{R}_X(t) \overset{\Delta}{=} \int_{\eta=0}^{t} R_X(\eta) d\eta = \sigma_X^2 \int_{\eta=0}^{t} e^{-\beta\tau} d\eta = \frac{\sigma_X^2}{\beta}\left(1 - e^{-\beta t}\right). \tag{12}$$

Hence, (8) is:

$$\breve{\breve{R}}_X(t) \overset{\Delta}{=} \int_{\eta=0}^{t} \breve{R}_X(\eta) d\eta = \frac{\sigma_X^2}{\beta} \int_{\eta=0}^{t} \left(1 - e^{-\beta\eta}\right) d\eta = \frac{\sigma_X^2}{\beta}\left[\eta + \frac{e^{-\beta\eta}}{\beta}\right]_{\eta=0}^{t} = \frac{\sigma_X^2}{\beta}\left[t + \frac{e^{-\beta t}}{\beta} - \frac{1}{\beta}\right]. \tag{13}$$

Appropriate substitution of (13) into (11) gives:

$$R_Y(t, t+\tau) = \frac{\sigma_X^2}{\beta}\left[t + \frac{e^{-\beta t}}{\beta} - \frac{1}{\beta}\right] + \frac{\sigma_X^2}{\beta}\left[t + \tau + \frac{e^{-\beta(t+\tau)}}{\beta} - \frac{1}{\beta}\right] - \frac{\sigma_X^2}{\beta}\left[\tau + \frac{e^{-\beta\tau}}{\beta} - \frac{1}{\beta}\right]. \tag{14}$$

This can be simplified to:

$$R_Y(t, t+\tau) = \frac{\sigma_X^2}{\beta}\left[2t - \frac{1}{\beta} + \frac{e^{-\beta t} + e^{-\beta(t+\tau)} - e^{-\beta\tau}}{\beta}\right]. \qquad (15).$$

In particular,

$$\sigma_{Y(t)}^2 = R_Y(t,t) = \frac{\sigma_X^2}{\beta}\left[2t - \frac{1}{\beta} + \frac{2e^{-\beta t} - 1}{\beta}\right] \quad = \frac{2\sigma_X^2}{\beta}\left[t + \frac{e^{-\beta t} - 1}{\beta}\right]. \tag{16}$$

For sufficiently small $\beta t$, since $e^{-\beta t} \cong 1 - \beta t$, (16) becomes $\sigma_{Y(t)}^2 \cong 0$, as expected. For sufficiently large $\beta t$ (16) becomes

$$\sigma_{Y(t)}^2 \quad \cong \quad \frac{2\sigma_X^2}{\beta}\left[t - \frac{1}{\beta}\right] = \frac{2\sigma_X^2}{\beta}\left[\frac{\beta t - 1}{\beta}\right] \cong \frac{2\sigma_X^2 t}{\beta} \quad \text{for} \quad \beta t >> 1. \tag{17}$$

More generally, from (15), for sufficiently large $\beta t$ we have

$$R_Y(t,t+\tau) = \frac{\sigma_X^2}{\beta}\left[\frac{2\beta t - e^{-\beta \tau} - 1}{\beta}\right] \cong \quad \frac{\sigma_X^2}{\beta}\left[\frac{2\beta t - e^{-\beta \tau}}{\beta}\right] \cong \frac{2\sigma_X^2 t}{\beta} \cong \sigma_{Y(t)}^2 \quad \text{for} \quad \beta t >> 1. \tag{18}$$

Equation (18) is interesting, in that it states that for any fixed *t*, (under the condition $\beta t >> 1$), the autocorrelation function $R_Y(t,t+\tau)$ is essentially constant- *independent* of $\tau$ ! This behavior is illustrated in the sample realizations shown below.

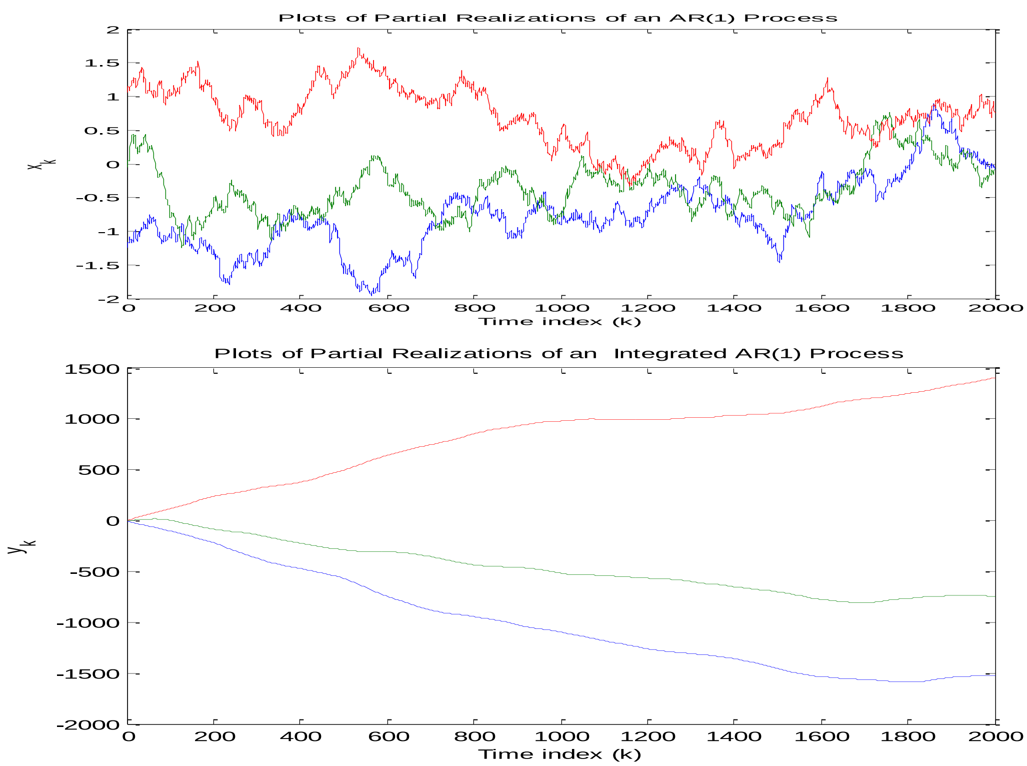


**Figure 1.** Three sample realizations of an AR(1) process (top) and its integral process (bottom).

The above figure clearly suggests that (i) $E[Y(t)] = 0$ for all *t*. The increasing variability illustrates the dependence of the variance (16) on *t*. The extreme smoothness illustrates that at any sufficiently large time, *t*, the product, $Y(t)Y(t+\tau)$ is, on the average, independent of the lag, $\tau$. Specifically, each of the realizations in Figure 1 suggests that from any chosen large *t*, $Y(t+\tau) \cong Y(t) + c\tau$. But the slope, *c*, is, on the average, zero. Hence, on the average, $Y(t+\tau) \cong Y(t)$.

Verification of the variance expression (17) is given in Figure 2.

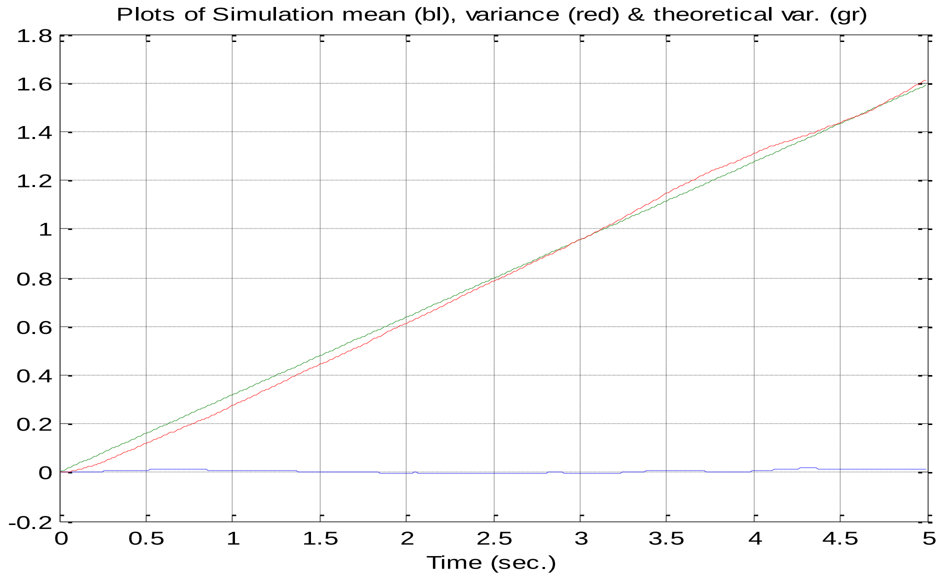


**Figure 2.** Simulation-based (n=2000) estimates of $E[Y(t)]$ (blue), and $\sigma^2_{Y(t)}$ (red). The theoretical variance given by (17) is shown in green.

The reader may feel that the above development was unduly pedantic. The reason for the attention to even the slightest details was the result of numerous unsuccessful attempts without sufficient care.

**The Practical Importance of the Above Results**

There are many situations in which the processes of interest include a given process and its integral. In particular, consider the situation where an accelerometer is use to estimate the velocity of an object. Specifically, suppose that the acceleration is governed by the following difference equation:

$$A_{k+1} = \alpha A_k + U_{k+1} \text{ where } \{U_k\} \sim iid\, N(0,\sigma_U^2) \tag{19a}$$

The velocity equation is then

$$V_{k+1} = V_k + A_k \Delta . \tag{19b}$$

Let the measurement equation be given as

$$Z_k = A_k + W_k \text{ where } \{W_k\} \sim iid\, N(0,\sigma_W^2) .$$

Let $X_k = [A_k\ V_k]^{tr}$ . Then the above equations admit the state space representation:

$$X_{k+1} = \begin{bmatrix} \alpha & 0 \\ \Delta & 1 \end{bmatrix} X_k \quad + \quad \begin{bmatrix} U_{k+1} \\ 0 \end{bmatrix} \tag{20a}$$

$$Z_k = \begin{bmatrix} 1 & 0 \end{bmatrix} X_k + W_k . \tag{20b}$$

The equations (20) are typical equations used in the application of Kalman filtering to estimate the velocity process, $V_k$ . Since $V_k$ is the 'integral' of a sampled *GM* process, it is a nonstationary process that has the properties of $Y(t)$ above. Specifically, it is smooth, and its variance grows at a rate proportional to *k*. Knowing this, one must ask: Is this what $V_k$ , in fact, behaves like?

For example, suppose that it is known that $V_k$ behaves like the AR(1) process:

$$V_{k+1} = \gamma V_k + Q_{k+1} \text{ where } \{Q_k\} \sim iid\ N(0,\sigma_Q^2) . \tag{21a}$$

If we approximate $A_k$ by the first order backward difference equation

$$A_{k+1} = \frac{V_{k+1} - V_k}{\Delta} \tag{21b}$$

then we have the following block diagram

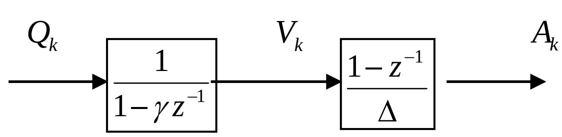


**Figure 2.** Block diagram associated with (21).

The difference equation that relates $A_k$ to $Q_k$ is

$$A_{k+1} = \gamma A_k + \Delta^{-1}(Q_{k+1} - Q_k)\,. \tag{22}$$

Hence, we see that if we believe that (21a) is a more reasonable model for $V_k$ than an integrated sampled *GM* process, then our model for $A_k$ should have the form (22), which is not an AR(1) model, but rather, an ARMA(1,1) model. In this case, the state equation (20a) becomes

$$X_{k+1} = \begin{bmatrix} \gamma & 0 \\ \Delta & 1 \end{bmatrix} X_k \quad + \quad \begin{bmatrix} \Delta^{-1} & -\Delta^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{k+1} \\ Q_k \end{bmatrix} \tag{23}$$

**Comment.** If one has conducted a thorough study of accelerometer measurements, and concluded that the AR(1) model, (19a) is the best model, then one must accept that the velocity has the properties of an integrated sampled *GM* process. Specifically, it must be acknowledged to be smooth, nonstationary, and have variability proportional to the sample time index, *k.* On the other hand, if the physics of the problem dictate that the velocity is *wss*, and can be represented by (21a), then the acceleration process must be specified accordingly.
[Note: I have encountered many situations, both in textbooks and as a committee member, where one desires to use accelerometer data to estimate velocity and/or position. I cannot recall a single case where any attention to the above has been given.]

Stated another way, suppose that we are interested in modeling a random process, $Y(t)$, and its derivative, $\dot{Y}(t)$, *both* as *wss* random processes. Then we must first model $Y(t)$, and then obtain the appropriate model for its derivative, $\dot{Y}(t)$. For, if we first model $\dot{Y}(t)$ as a *wss* process, then the appropriate model for $Y(t)$ will *not* be *wss.*

To appreciate this in relation to rational transfer function models, suppose that our model for $\dot{Y}(t) \stackrel{\Delta}{=} X(t)$ as a *GM* process is:

$$X(s) = \left[\frac{1}{s+\beta}\right] U(s) \tag{24a}$$

where $U(t)$ is a fictitious white noise process. Then the appropriate model for $Y(t)$ is:

$$Y(s) = \left[\frac{1}{s(s+\beta)}\right] U(s)\,. \tag{24b}$$

However, because the transfer function in (24b) has a pole on the imaginary axis (in this case at zero), the system is *unstable*. Hence, $Y(t)$ will *not* be *wss.* On the other hand, if we model $Y(t)$ using the transfer function in (24a), then its derivative, $\dot{Y}(t) \overset{\Delta}{=} X(t)$, has the model:

$$X(s) = \left[ \frac{s}{s+\beta} \right] U(s) . \tag{24c}$$

Even though the transfer function in (24c) is not a *proper* one, it is still, nonetheless, a stable system. Hence, the process $\dot{Y}(t)$ will be a *wss* process.

**Remark.** The above material is, by no means, mathematically rigorous; especially in relation to the mathematics used by experts in the area. For example, while (24) appears reasonable enough on the surface, the corresponding stochastic differential equation is:

$$\dot{X}(t) + \beta X(t) = \dot{U}(t) . \tag{25}$$

In (25) the 'input' is not a fictitious white noise process, but rather the derivative of such a process.

**Appendix** ***Matlab*** **code used to generate Figure 1.**

```
% Program name: sumar1.m
%=====================================================
nsim = 2000; % Number of simulations
m = 5;       % Number of simulations to be plotted
%=====================================================
% Sampling Specifications
n = 1000;       % Length of partial realization (integer)
fs = 200;      % Sampling Frequency (Hz)
dt = 1/fs;     % Sampling Period (sec/sample)
T = n*dt;      % Observation window (seconds)
t = 0:dt:T-dt; % Time array
%======================================================
% Continuous & Discrete Process PARAMETERS
b = 1*(2*pi);    % -3dB BW (rad/sec) = 1 HZ
a=exp(-b*dt);    % BW parameter for sampled process
se = (1-a^2)^.5; % white noise std for varx = 1
%=======================================================
% Simulations
e = zeros(nsim,n); x = e; y = e; % Initialize arrays
e = se*randn(nsim,n);
x(:,1) = randn(nsim,1); y(:,1) = x(:,1);
for k = 2:n
    x(:,k) = a*x(:,k-1) + e(:,k);
    y(:,k) = y(:,k-1) + x(:,k);
end
y = dt*y;
%=======================================================
% FIGURES 1 & 2: Plots of m Simulations of the AR(1)& IAR Processes
figure(1)
tplot = []; xplot = []; yplot = [];
for mm = 1:m
    tplot = [tplot ; t];
    xplot = [xplot ; x(mm,:)];
    yplot = [yplot ; y(mm,:)];
end
tplot = tplot'; xplot = xplot'; yplot = yplot';
plot(tplot,xplot)
xlabel('Time (sec.)')
ylabel('x_k')
title('Plots of Partial Realizations of an AR(1) Process')
grid
pause
figure(2)
plot(tplot,yplot)
xlabel('Time (sec.)')
ylabel('y_k')
title('Plots of Partial Realizations of an  Integrated AR(1) Process')
grid
pause
%========================================================
% Sample Statistics for the Integrated Process
my =  mean(y); my = my';        % Simulation-based Mean of Y(t)
vary = var(y); vary = vary';    % Simulation-based Variance of Y(t)
varyth = (2/b)*t';               % Theoretical (large-t) Variance of Y(t)
```

```
mvplot = [my varyth vary];
figure(3)
tplot3 = tplot(:,1:3);
plot(tplot3,mvplot)
xlabel('Time (sec.)')
title('Plots of Simulation mean (bl), variance (red) & theoretical var.
(gr)')
grid
```