Lecture 16 *m*-Step-Ahead Prediction

Consider the time-varying single-input/single-output (SISO) system:

$$x_{k+1} = \alpha_k x_k + f_{k+1}. \tag{1}$$

We will now develop the formula for x_{k+m}.

$$x_{k+2} = \alpha_{k+1} x_{k+1} + f_{k+2} = \alpha_{k+1}(\alpha_k x_k + f_{k+1}) + f_{k+2} = \alpha_{k+1}\alpha_k x_k + (\alpha_{k+1} f_{k+1} + f_{k+2}).$$

$$x_{k+3} = \alpha_{k+2}\alpha_{k+1}\alpha_k x_k + (\alpha_{k+2}\alpha_{k+1} f_{k+1} + \alpha_{k+2} f_{k+2} + f_{k+3}).$$

$$x_{k+m} = \alpha_{k+m-1}\cdots\alpha_{k+2}\alpha_{k+1}\alpha_k x_k + [(\alpha_{k+m-1}\cdots\alpha_{k+2}\alpha_{k+1}) f_{k+1} + (\alpha_{k+m-1}\cdots\alpha_{k+2}) f_{k+2} + \cdots + f_{k+m}].$$

This can be expressed in a more compact form as:

$$x_{k+m} = \left(\prod_{j=0}^{m-1}\alpha_{k+m-1-j}\right)x_k + \left[\left(\prod_{j=0}^{m-2}\alpha_{k+m-1-j}\right) f_{k+1} + \left(\prod_{j=0}^{m-3}\alpha_{k+m-1-j}\right) f_{k+2} + \cdots + f_{k+m}\right]. \tag{2}$$

To gain a better idea of the structure of (2), consider the case where $\alpha_k = \alpha$. Then (2) becomes

$$x_{k+m} = \alpha^m x_k + \left[\alpha^{m-1} f_{k+1} + \alpha^{m-2} f_{k+2} + \cdots + f_{k+m}\right] = \alpha^m x_k + \sum_{j=0}^{m-1}\alpha^{m-1-j} f_{k+1+j}. \tag{3}$$

Equation (3) includes two terms. The first is $\alpha^m x_k$, which is the response to the 'initial condition' x_k. The second is $\sum_{j=0}^{m-1}\alpha^{m-1-j} f_{k+1+j}$, which is the response to the input $\left\{ f_{k+j} \right\}_{j=1}^{m}$.

We are now in a position to readily address the *m*-step-ahead KF prediction problem that includes a deterministic input. Specifically, consider the 1-step ahead predictor:

$$\hat{x}_{k+1} = \alpha_k \hat{x}_k + f_{k+1}. \tag{4}$$

From (2), the m step-ahead predictor is:

$$\hat{x}_{k+m} = \left(\prod_{j=0}^{m-1}\alpha_{k+m-1-j}\right)\hat{x}_k + \left[\left(\prod_{j=0}^{m-2}\alpha_{k+m-1-j}\right) f_{k+1} + \left(\prod_{j=0}^{m-3}\alpha_{k+m-1-j}\right) f_{k+2} + \cdots + f_{k+m}\right]. \tag{5}$$

Even though the above results were addressed in the SISO setting, there is absolutely no difference if one views them in the MIMO setting. Notationally, this amounts to:

$$\hat{\mathbf{x}}_{k+m} = \left(\prod_{j=0}^{m-1}\mathbf{A}_{k+m-1-j}\right)\hat{\mathbf{x}}_k + \left[\left(\prod_{j=0}^{m-2}\mathbf{A}_{k+m-1-j}\right)\mathbf{f}_{k+1} + \left(\prod_{j=0}^{m-3}\mathbf{A}_{k+m-1-j}\right)\mathbf{f}_{k+2} + \cdots + \mathbf{f}_{k+m}\right]. \tag{6}$$

The only care that must be exercised in relation to (6) is the order in which the matrices are multiplied. For example, the state transition matrix that takes the state $\hat{\mathbf{x}}_k$ to the state $\hat{\mathbf{x}}_{k+m}$ in the absence of any input is

$$\mathbf{A}(k+m,k)=\prod_{j=0}^{m-1}\mathbf{A}_{k+m-1-j}=\mathbf{A}_{k+m-1}\mathbf{A}_{k+m-2}\cdots\mathbf{A}_k \, .$$

As noted on pp.159-160 in the book, the associated prediction error is:

$$\mathbf{P}(k+m\,|\,k)=\mathbf{A}(k+m,k)\mathbf{P}_k\mathbf{A}(k+m,k)^{tr}+\mathbf{Q}(k+m,k) \, .$$

Remark concerning Exam 3:

In Exam 3 PROBLEM 2(e) it is stated: Let the *m*-step ahead predictor of $\mathbf{x}_{t+m}$ be $\hat{\mathbf{x}}_{t+m}=\mathbf{F}_t^m\hat{\mathbf{x}}_t$. Comparing this to (5), it is clear that this is not the best predictor, as it ignores the deterministic input. The reason that this suboptimal predictor was given to you is twofold. First, it is clear that (5) is significantly more complicated. Since we have not covered prediction to this point, I felt it was unfair to expect you to compute (5), much less arrive at it. The second reason is that it turns out that for $m=5$, you should find that this suboptimal predictor performs reasonably well.

The prediction result for my simulation is shown at right.

While it would be instructive to compare it to the prediction using (5), I do not believe that such an exercise would be in the best interests of students at this point. The frustration would almost surely outweigh the gain in any knowledge.

My intent was to make you aware of the predictive capacity of the KF in a relatively painless manner. All students should be quite capable of computing $\hat{\mathbf{x}}_{t+m}=\mathbf{F}_t^m\hat{\mathbf{x}}_t$.

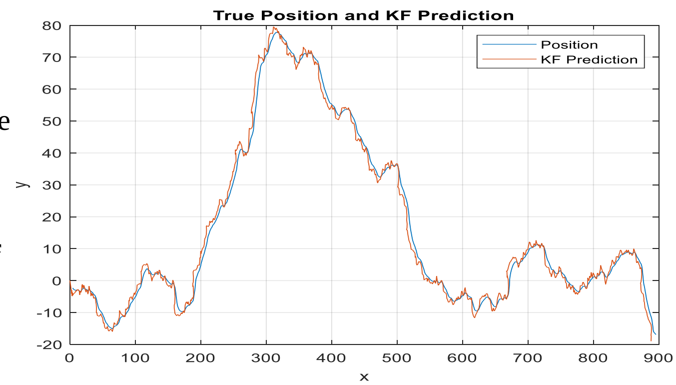

Figure 2(d) Plots of $(\mathbf{x}_{t+5},\mathbf{y}_{t+5})$ and $(\hat{\mathbf{x}}_{t+5},\hat{\mathbf{y}}_{t+5})$.