Lecture 15 **More on Kalman Filtering and the MMAE**

In these notes we will go into more detail in relation to the following problem. An AR(1) process can have one of two possible bandwidth parameters $\{\alpha_j\}_{j=1}^2$. The parameter α_1 corresponds to a process associated with a normal condition, and α_2 corresponds to an abnormal condition. We will make the following assumptions:

(A1): When the process transitions to the abnormal condition, it remains
so for a period of time.

(A2): The process power σ_z^2 is the same, regardless of the condition.

An example of such a measured process is shown in Figure 1. It was derived from a Gauss-Markov process with $\sigma_z^2 = 1$, $f_{bw}^{(1)} = 1Hz$, $f_{bw}^{(2)} = 3Hz$, and $f_{samp} = 50\,Hz$. The corresponding AR(1) BW parameters are $\alpha_1 = 0.88$ and $\alpha_2 = 0.69$.

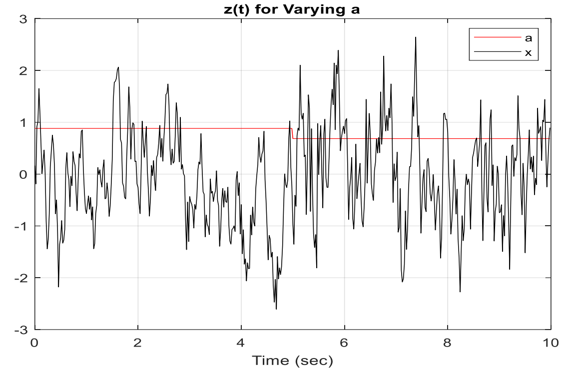


Figure 1. Measurement of z_k.

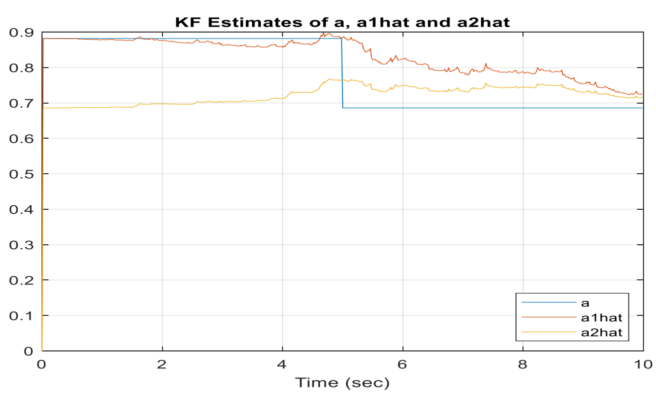
The Kalman filter (KF) multiple model adaptive estimation (*mmae*) method uses two KFs run in parallel. The state and measurement equations are:

$$\alpha_{k+1,j} = \alpha_{k,j} + u_{k,j} \quad \text{for } j = 1,2 \quad (1a) \qquad ; \qquad z_k = \alpha_j z_{k-1} + v_{k,j} \quad (1b)$$

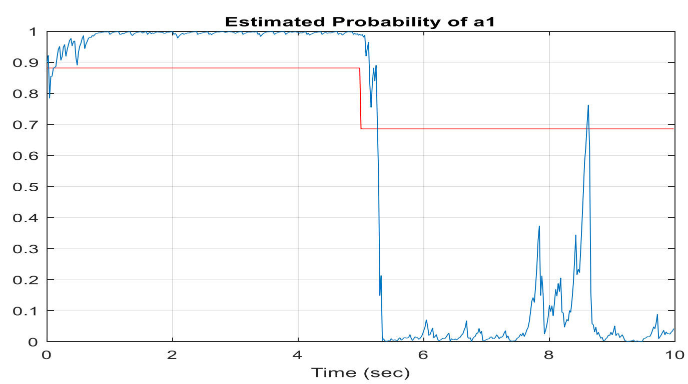
where $p = \Pr[\alpha = \alpha^{(1)}]$, and where $\sigma_{v,j}^2 = \sigma_z^2 \sqrt{1-\alpha_j^2}$. From the Lecture 14 notes, we have:

$$\Pr[A = a_1 \mid \hat{z}_k^-] = \frac{f(z_k - \hat{z}_k^- \mid a_1)p}{f(z_k - \hat{z}_k^- \mid a_1)p + f(z_k - \hat{z}_k^- \mid a_2)(1-p)}. \qquad (2)$$

It should be noted that in (2) there are two different $\hat{z}_k^-$ quantities; one associated with each KF. The KF estimates of $\{\alpha_j\}_{j=1}^2$, and the estimated probability (2) are given in Figure 2.



(a)



(b)

Figure 2. (a) KF estimates of $\{\alpha_j\}_{j=1}^2$, and **(b)** estimated probability (2).

From Figure 2(a) it should be clear that the state variances $\{Q_j\}_{j=1}^2$ were chosen to be quite small. After the transition from the normal to the abnormal state, both state estimates are slow to track the change. It is for this reason that the estimated probabilities in the abnormal state are a bit variable. Because we began the simulation in the normal state, and assigned a probability of 0.9 to it, there is little variability in the estimated probability in the normal region.

It should be noticed that time increases, the KF that uses the i.c. α_2 begins to move toward α_1. This can be expected for any nonzero driving variance in the random walk. Similar behavior is evident in the abnormal region. The KF estimate that uses the i.c. α_1 moves close and closer to α_2; thereby influencing the probability estimate. On could avoid both scenarios by setting the Q-values to zero. The price paid would be that one would then have no tracking information re: α.

We will now proceed to review the associated Matlab code in detail. We will see that a number of issues needed to be addressed.

```
% PROGRAM NAME: eegmmae.m
%Simulation of eeg:
fbw = [1 ; 3]; %BW (Hz) for two regions
varZ=[1;1]; %Variances for two regions
fs=50; %Sampling frequency (Hz)
del=1/fs;
a=exp(-2*pi*fbw*del); %AR(1) BW parameters
varV=varZ.*(1-a.^2); %Measurement driving variances
T=10; %Total observation time (sec)
npts=fix(T/del);
t=0:del:(npts-1)*del;
A=[a(1)*ones(1,npts/2) , a(2)*ones(1,npts/2)];
stdv=[sqrt(varV(1))*ones(1,npts/2) , sqrt(varV(2))*ones(1,npts/2)];
z=zeros(1,npts);
z(1)=normrnd(0,sqrt(varZ(1)));
for k=2:npts
    z(k)=A(k)*z(k-1)+normrnd(0,stdv(k));
end
figure(1)
plot(t,A,'r')
hold on
plot(t,z,'k')
title('z(t) for Varying a')
xlabel('Time (sec)')
legend('a','x')
grid
%***********************************************************
%KF:
% Driving noise variances for AR(1) processes
R=[varV(1)*ones(1,npts);varV(2)*ones(1,npts)];
Q=[.00001 ; .00001]; %ARbw parameter state driving variances
%COMMENT 1: Ideally, these should be found by minimizing each mse.
%=========================================================
% PROBABILITY INITIAL CONDITIONS
pa0=[0.9;0.1]; %Assign prior probabilities to a1 & a2
pm1=pa0;
%=========================================================
% KALMAN FILTER INITIAL CONDITIONS
xhat=zeros(2,npts);
xhatm=[a(1);a(2)];
Pm=Q;
I=ones(2,1);
Pra1=zeros(1,npts);
Pra1(1)=0.9;
%--------------------
for k=2:npts
   H=z(k-1)*[1;1];
   Kk= Pm.*H.*(H.*Pm.*H + R(:,k)).^-1;
   zm=H.*xhatm;
   xhat(:,k)=xhatm + Kk.*(z(:,k)-zm);
   P=(I-Kk.*H).*Pm;
   xhatm=xhat(:,k);
   Pm=P + Q;
   %----------------------
   f1=normpdf(z(k),zm(1),sqrt(H(1)^2*Pm(1)+R(1,k)));
   f2=normpdf(z(k),zm(2),sqrt(H(2)^2*Pm(2)+R(2,k)));
   p1=f1*Pra1(k-1);
   p2=f2*(1-Pra1(k-1));
   Pra1(k)=p1/(p1+p2);
   if Pra1(k)>0.999
      Pra1(k)=0.99;
   elseif Pra1(k) <.001
```

```
        Pra1(k)=0.01;
    end
end
figure(2)
plot(t,[A;xhat])
title('KF Estimates of a, a1hat and a2hat')
xlabel('Time (sec)')
legend('a','a1hat','a2hat','Location','SouthEast')
grid
figure(3)
plot(t,A,'r')
hold on
plot(t,Pra1)
title('Estimated Probability of a1')
xlabel('Time (sec)')
grid
```