Lecture 9 **Review of Linear Systems**

We will motivate this topic by connecting the concept of a linear system to the Wiener filter example in Lecture 14. To this end, we must first define what a linear system is.

Definition 1. A *linear system* is a **relation** between two time-varying quantities. Denote the **input** to the system as $u(t)$ and denote the **output** as $y(t)$. A relation between $u(t)$ and $y(t)$ of the form

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)u(\tau)d\tau \tag{1}$$

defines the linear system $h(t,\tau)$. If $h(t,\tau) = h(t-\tau)$, the system is called a *time-invariant linear system.* Furthermore, if the upper limit of integration is *t*, then the system is called a *causal* system.

In this lecture we will focus on time-invariant causal linear systems. In this case, (1) becomes:

$$. \; y(t) = \int_0^t h(t-\tau)u(\tau)d\tau \equiv \int_0^t h(\tau)u(t-\tau)d\tau \; . \tag{2}$$

The rightmost equality in (2) follows from the change of variable theorem. The relation (2) is called a *convolution integral.*

Now consider the input $u(t) = \delta(t)$, which is a unit impulse (i.e. a Dirac delta function). Then

$$y(t) = \int_0^t h(t-\tau)\delta(\tau)d\tau = h(t) \; . \tag{3}$$

Since the output $y(t)$ is the response to an impulse, and since $y(t) = h(t)$, the quantity $h(t)$ is called the system *impulse response function.*

Definition 2. The *Laplace transform* of $y(t)$ defined on $[0,\infty)$ is $Y(s)) \overset{\Delta}{=} \int_{t=0}^{\infty} y(t)e^{-st}dt$ for $s = \sigma + i\omega$. For $s = i\omega$, the function $Y(i\omega)$ is called the *Fourier transform* of $y(t)$.

Example 1. [See also Lecture 14 Example.] Consider the following difference equation models for a signal process s_k and noise process n_k:

$s_k = 0.9\, s_{k-1} + u_k$; $n_k = -0.5\, n_{k-1} + v_k$, where u_k and v_k are white noise processes.

(a) Suppose that the difference equation $s_k = 0.9\, s_{k-1} + u_k$ was arrived at by sampling a continuous-time signal $s(t)$. We claim that $s(t) = \int_0^t h(\tau)u(t-\tau)d\tau$ where $h(t) = e^{-\beta t}$. To show this, we will approximate the input as

$u(t) = \delta(t) \cong (1/\Delta)\delta_K(n\Delta)$ where $\delta_K(n\Delta) = \begin{cases} 1 \ \ for\ n = 0 \\ 0 \ \ for\ n \neq 0 \end{cases}$ is called the *Kronecker delta function.* The Riemann sum approximation of the above integral is:

$$s(n\Delta) = \sum_{k=0}^{n} h(k\Delta)(1/\Delta)\delta_K(n\Delta - k\Delta)\Delta = h(n\Delta) = e^{-\beta(n\Delta)}.$$

In words, $s(n\Delta)$ is the response to any input $u(n\Delta)$, and the system impulse response is $h(n\Delta) = e^{-\beta(n\Delta)}$. Define the parameter $\alpha \overset{\Delta}{=} e^{-\beta\Delta}$. Then $h(n\Delta) = e^{-\beta(n\Delta)} = \alpha^n$. Hence, for any input $u(n\Delta) \overset{\Delta}{=} u_n$, we have $s_n = \sum_{k=0}^{n} \alpha^k u_{n-k}$. However, this can be written as:

$$s_n = \sum_{k=0}^{n} \alpha^k u_{n-k} = \sum_{k=1}^{n} \alpha^k u_{n-k} + u_n = \alpha \sum_{k=0}^{n-1} \alpha^{k-1} u_{n-(k-1)} + u_n = \alpha s_{n-1} + u_n. \tag{4}$$

The relation $s_n = \alpha s_{n-1} + u_n$ is exactly the given relation $s_k = 0.9\, s_{k-1} + u_k$ for $\alpha = 0.9$.

The above example entailed a fair bit of mathematics. Fortunately, we will be able to skirt a lot of such mathematics once we have a few more properties associated with linear systems. The following are some of the most important properties.

Property 1. [***The convolution theorem***]. The Laplace transform of the convolution equation $y(t) = \int_0^t h(t-\tau)u(\tau)d\tau$ is $Y(s) = H(s)U(s)$.

Proof: Because both $h(t)$ and $u(t)$ are *causal* functions of time (i.e. they are both zero for $t < 0$, we can write $y(t)$ as: $y(t) = \int_{\tau=0}^{\infty} h(t-\tau)u(\tau)d\tau$. Taking the Laplace transform of this expression gives:

$$Y(s) = \int_{t=0}^{\infty} y(t)e^{-st}dt = \int_{t=0}^{\infty}\int_{\tau=0}^{\infty} h(t-\tau)u(\tau)e^{-s(t-\tau)}e^{-s\tau}d\tau dt.$$

For any chosen τ, define the variable $v = t - \tau$. Then $dv = dt$, so that the above equation becomes:

$$Y(s) = \int_{v=0}^{\infty}\int_{\tau=0}^{\infty} h(v)u(\tau)e^{-sv}e^{-s\tau}d\tau dv = \int_{v=0}^{\infty} h(v)e^{-sv}e^{-sv}dv \int_{\tau=0}^{\infty} u(\tau)e^{-s\tau}d\tau = H(s)U(s). \quad \square$$

This is most important property in relation to linear time-invariant systems. It states that ***convolution in the time domain is equivalent to multiplication in the s-domain.***

From Property 1, we have $H(s) = Y(s)/U(s)$.

Definition 3. The system *transfer function* is $H(s) = Y(s)/U(s)$.

Property 2. The system transfer function $H(s)$ is the Laplace transform of the system *impulse response function* $h(t)$.

Proof: For $u(t) = \delta(t)$, $U(s) = \int_{t=0}^{\infty} \delta(t)e^{-st}dt = 1$. Hence, $Y(s) = H(s)U(s)$ becomes $Y(s) = H(s)$. □

A word of caution here. Even though mathematically we have $Y(s) = H(s)$, the units in this equality do not match. The units of $Y(s)$ are the units of the output $y(t)$, whereas the units of $H(s)$ are the units of $y(t)$ divided by those of $u(t)$.

The system transfer function is the s-domain relation between the input and output for *any* input. This is why transfer functions are so powerful. They describe the properties of the system, itself.

Property 3. Let $y(t) \leftrightarrow Y(s)$. Then $\dot{y}(t) \leftrightarrow sY(s) - y_0$.

Proof: By definition, the Laplace transform of $\dot{y}(t)$ is $\int_{t=0}^{\infty} \dot{y}(t)e^{-st}dt$. Let $u = e^{-st}$ and $dv = \dot{y}(t)dt$. Then

$du = -se^{-st}$ and $v = y(t)$, so that integration by parts gives:

$\int_{t=0}^{\infty} \dot{y}(t)e^{-st}dt = \int udv = uv - \int vdu = y(t)e^{-st}\Big|_{t=0}^{\infty} + s\int_{t=0}^{\infty} y(t)e^{-st}dt = 0 - y(0) + sY(s)$. □

Using Property 3, we have: $g(t) = \dot{y}(t) \leftrightarrow G(s) = s^2Y(s) - sy_0 - \dot{y}_0$. This follows directly from the fact that $\dot{g}(t) = \ddot{y}(t) = \leftrightarrow sG(s) - g_0 = s\left[sY(s) - y_0\right] - \dot{y}_0$. Hence, we can generalize Property 2 as

Property 3'. Let $y(t) \leftrightarrow Y(s)$. Then $y^{(m)}(t) \leftrightarrow s^mY(s) - s^{m-1}y_0 - \ldots - y_0^{(m-1)}$, where $y^{(m)}(t) \stackrel{\Delta}{=} d^{(m)}y(t)/dt^m$.

Notice that is all initial conditions are zero, then $y^{(m)}(t) \leftrightarrow s^mY(s)$. This is the case in the definition of the system transfer function associated with a linear constant-coefficient differential equation.

Property 4. Consider the system described by: $a_ny^{(n)}(t) + \ldots + a_1y^{(1)}(t) + a_0y(t) = b_mu^{(m)}(t) + \ldots + b_1u^{(1)}(t) + b_0u(t)$, and assume that all $n-1$ initial conditions are zero. Since the forcing input $u(t)$ has, by definition, no initial conditions, the Laplace transform of the equation is: $a_ns^nY(s) + \ldots + a_1sY(s) + a_0Y(s) = b_ms^mU(s) + \ldots + b_1sU(s) + b_0U(s)$.

Hence, the system transfer function is: $H(s) = Y(s)/U(s) = \dfrac{b_ms^m + \ldots + b_1s + b_0}{a_ns^n + \ldots + a_1s + a_0}$.

Example 1 continued. For impulse response $h(t) = e^{-\beta t}$, the system transfer function is:

$$H(s) = \int_{t=0}^{\infty} h(t)e^{-st}dt = \int_{t=0}^{\infty} e^{-\beta t}e^{-st}dt = \int_{t=0}^{\infty} e^{-(s+\beta)t}dt = \frac{e^{-(s+\beta)t}}{-(s+\beta)}\Bigg|_{t=0}^{\infty} = \frac{1}{s+\beta} = \frac{Y(s)}{U(s)}.$$

Hence, a system with this impulse response can be written as the differential equation: $\dot{y}(t) + \beta y(t) = u(t)$. Recall that a Gauss-Markov process is a process that has an autocorrelation function of the form $R(\tau) = \sigma_y^2 e^{-\beta|\tau|}$. The *psd* for such a process is :

$$S_y(\omega) \overset{\Delta}{=} \int_{-\infty}^{\infty} R_y(\tau) e^{-i\omega\tau} d\tau = \sigma_y^2 \int_{-\infty}^{\infty} e^{-i\omega\tau} e^{-\beta|\tau|} d\tau = \sigma_y^2 \int_0^{\infty} e^{-\beta\tau} e^{-i\omega\tau} d\tau + \sigma_y^2 \int_{-\infty}^{0} e^{\beta\tau} e^{-i\omega\tau} d\tau =$$

$$\sigma_y^2 \int_0^{\infty} e^{-\beta\tau} e^{-i\omega\tau} d\tau + \sigma_y^2 \int_0^{\infty} e^{-\beta\tau} e^{i\omega\tau} d\tau = \sigma_y^2 \left(\frac{1}{\beta + i\omega} + \frac{1}{\beta - i\omega} \right) = \frac{2\beta\sigma_y^2}{\beta^2 + \omega^2}.$$

Define the fictitious system transfer function $H(s) = \frac{1}{s+\beta} = \frac{Y(s)}{U(s)}$ where the fictitious input is white noise. Then $Y(i\omega) = \frac{U(i\omega)}{\beta + i\omega}$, so that from the Wiener-Kinchin Theorem we have:

$S_y(\omega) = E[|Y(i\omega)|^2] = \frac{E[|U(i\omega)|^2]}{\beta^2 + \omega^2} = \frac{c}{\beta^2 + \omega^2}$ where the white noise *psd* is $S_u(\omega) = E[|U(i\omega)|^2] = c$. If we set $c = 2\beta\sigma_y^2$, we see that the GM process $y(t)$ can be viewed as the output of a system with transfer function $H(s) = \frac{1}{s+\beta}$ that is excited by a fictitious white noise input with *psd* $S_u(\omega) = 2\beta\sigma_y^2$. □

Property 5. [The time delay theorem.] Let $y(t) \leftrightarrow Y(s)$. Then $y(t - t_0) \leftrightarrow e^{-st_0} Y(s)$.

Proof: $\int_{t=0}^{\infty} y(t - t_0) e^{-st} dt = e^{-st_0} \int_{t=t_0}^{\infty} y(t - t_0) e^{-s(t-t_0)} dt = e^{-st_0} \int_{\tau=0}^{\infty} y(\tau) e^{-s\tau} d\tau = e^{-st_0} Y(s)$. □

SUMMARY

The above was a brief but dense introduction to time-invariant single input-single output linear systems and how they relate to *wss* random processes. Important properties of Laplace transform pairs were given. In relation to these, some proofs were included. This is not a course in mathematics. The goal of including proofs was to illustrate how material in past calculus courses relates to the material at hand. Hopefully, this will give the student more confidence in understanding the material, as opposed to simply memorizing it. For our purposes, the most important result is that often a *wss* random process can be associated with a differential equation having a real 'output', but a fictitious white noise 'input'. Given such a model, the notion of a transfer function allows one to use multiplication in the s-domain, as opposed to convolution in the time-domain. Finally, when setting $s = i\omega$ the transfer function becomes the system *frequency response function* (*FRF*). This allows one to view the problem in the frequency domain, which often can give much more insight than viewing the problem in the time domain.