Transfer Functions & AERE355

In this set of notes we address the second order, constant coefficient differential equation:

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{f}(t) + b_o f(t) \quad ; y(0) = y_0 \quad \& \quad \dot{y}(0) = \dot{y}_0 . \tag{1}$$

This equation describes the relation between an *input* $f(t)$ and an *output* $y(t)$. We will restrict our attention to the solution of (1) via Laplace transforms. To this end, we will need the following:

Definition 1: $Y(s) = \int_0^\infty y(t)e^{-st}dt \stackrel{\Delta}{=} \ell[y](s)$

Fact 1: $\ell[\dot{y}](s) = sY(s) - y_0$.

<u>*Proof*</u>: $\ell[\dot{y}](s) \stackrel{\Delta}{=} \int_0^\infty \dot{y}(t)e^{-st}dt$. Let $u = e^{-st}$ and $dv = \dot{y}(t)dt$. Then $du = -se^{-st}$ and $v = y(t)$.

Hence, $\ell[\dot{y}](s) \stackrel{\Delta}{=} \int_0^\infty u dv = uv\Big|_{t=0}^{\infty} - \int_0^\infty v du = y(t)e^{-st}\Big|_{t=0}^{\infty} + s\int_0^\infty y(t)e^{-st}dt = -y_0 + sY(s)$. □

Fact 2: $\ell[\ddot{y}](s) = s^2Y(s) - sy_0 - \dot{y}_0$.

<u>*Proof*</u>: Let $g(t) \stackrel{\Delta}{=} \dot{y}(t)$. Then from Fact 1, $\ell[\ddot{y}](s) = \ell[\dot{g}](s) = sG(s) - g_0 = s[sY(s) - y_0] - \dot{y}_0 = s^2Y(s) - sy_0 - \dot{y}_0$ □

Using the above, we will now use Laplace transforms to solve (1) for $Y(s)$:

$\ell[a_2 \ddot{y} + a_1 \dot{y} + a_0 y] = \ell[b_1 \dot{f}(t) + b_o f(t)]$ gives:

$a_2[s^2Y(s) - sy_0 - \dot{y}_0] + a_1[sY(s) - y_0] + a_0Y(s) = b_1sF(s) + b_oF(s)$. Rearranging gives:

$(a_2s^2 + a_1s + a_0)Y(s) -] + a_1[sY(s) - y_0] + a_0Y(s) = b_1sF(s) + b_oF(s) + a_2(sy_0 - \dot{y}_0) + a_1y_0$. Hence:

$$Y(s) = \left(\frac{b_1s + b_o}{a_2s^2 + a_1s + a_0}\right)F(s) + \frac{a_2\dot{y}_0s + a_1y_0}{a_2s^2 + a_1s + a_0} \stackrel{\Delta}{=} H(s)F(s) + Y_{ic}(s) = Y_f(s) + Y_{ic}(s) . \tag{2}$$

We see that $Y(s)$ is the sum of $Y_f(s) = H(s)F(s)$ associated with the input $f(t)$ plus $Y_{ic}(s)$ associated with the initial conditions. This lead to the definition of a system transfer function.

Definition 2. For a linear constant coefficient differential equation with input $f(t)$ and output $y(t)$, the system *transfer function* is $H(s) = Y(s)/F(s)$ assuming zero initial conditions.

Furthermore, <u>we will assume here that all initial conditions are zero</u>. The reader will address the case in which they are not in the homework problems. Taking the Laplace transform of (1), we obtain: $(a_2s^2 + a_1s + a_0)Y(s) = (b_1s + b_o)F(s)$ **.** It follows that the system *transfer function* is:

$$H(s) \stackrel{\Delta}{=} \frac{Y(s)}{F(s)} = \frac{b_1s + b_o}{a_2s^2 + a_1s + a_0} . \tag{3}$$

For any *input* $f(t)$ the solution of (1) under zero initial conditions is, in the *s*-domain, $Y(s) = H(s)F(s)$ **.**

The Unit Impulse Response of a Second Order System- For $f(t) = \delta(t)$ we have $Y(s) = H(s)$ **.** Notice that the units of (2) is the ratio of the output to the input. Hence, while $Y(s) = H(s)$ is mathematically correct, one must note that the units of

$Y(s) = H(s)$ as the solution for a unit impulse are those of the output. This relation is of such fundamental importance that we highlight it as

An Important Result- The system impulse response and transfer function constitute a Laplace transform pair. Even so, the transfer function units are those of the output divided by those of the input, while the impulse response units are those of the output alone.

Conditions for a Stable System-

The system (1) is a *stable* system if the impulse response $h(t) \to 0$ as $t \to \infty$. We will now show that this requires that the two roots of the *characteristic polynomial* $p(s) = a_2 s^2 + a_1 s + a_0$ both be in the *Left Half Plane* (*LHP*). To this end, recall that the *assumed solution* for the homogeneous O.D.E. has the form:

$$y_h(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} . \tag{4}$$

<u>Case 1</u>: Two real roots. In this case both s_1 and s_2 must be negative in order for (4) to go to zero as $t \to \infty$. Furthermore, since $y_h(t)$ is *real-valued*, both C_1 and C_1 must be *real* numbers.

<u>Case 2</u>: Complex-conjugat roots. In this case, $s_1 = \sigma + i\omega$ and $s_2 = \sigma - i\omega$. Hence, (4) becomes: $y_h(t) = e^{\sigma t}\left(C_1 e^{i\omega t} + C_2 e^{-i\omega t}\right)$. Furthermore, since $y_h(t)$ is *real-valued*, we must have $C_1 =| C_1 | e^{i\theta}$ and $C_2 =| C_1 | e^{-i\theta}$.

Hence:

$$y_h(t) =| C_1 | e^{\sigma t}\left(e^{i(\omega t+\theta)} + e^{-i(\omega t+\theta}\right) = 2 | C_1 | e^{\sigma t} \cos(\omega t + \theta) . \tag{5}$$

It should be clear that in order for (5) to go to zero as $t \to \infty$, we must have $\sigma < 0$.

Notice that this polynomial is the denominator of $H(s)$ in (3). Hence, the roots of $p(s)$ are the values of *s* that make $H(s)$ infinite.

Definition 3. For a transfer function $H(s) = n(s)/p(s)$, the system *poles* are the roots of $p(s)$ and the system *zeros* are the roots of $n(s)$.

Hence, we have the following important result.

***Result* 1.** A system $H(s)$ will be stable if and only if all the system poles are in the proper *LHP*.

The responses (4) and (5) are structurally different. While both decay exponentially, (5) oscillates as it decays.

Definition 3. A second order stable system having two real roots is called an ***overdamped*** system. If the roots are real and equal, it is called a ***critically damped*** system. If the roots are a complex-conjugate pair, then it is called an ***underdamped*** system.

This definition suggests that there is a parameter, call it ζ , that measures the amount of damping. Indeed, there is. To this end, write $s_{1,2} = \sigma \pm i\omega \overset{\Delta}{=} -\zeta\omega_n \pm i\omega_d$, where $\omega_d \overset{\Delta}{=} \omega_n\sqrt{1-\zeta^2}$. The parameter ζ must take on values $\zeta > 0$ for the system to be stable; for were we to have $\zeta = 0$, the poles would be purely imaginary (i.e. not in the proper *LHP*). Also, in order to has complex-conjugate roots, we must have $\zeta < 0$.

We can now write (5) as:

$$y_h(t) =| C_1 | e^{-\zeta\omega_n t}\cos(\omega_d t + \theta) \overset{\Delta}{=} | C_1 | e^{-t/\tau}\cos(\omega_d t + \theta) . \tag{6}$$

The parameter $\tau = 1/\zeta\omega_n$ is called the system *time constant*. The response (6) will be within 2% of zero at 4τ. Hence, people often refer to the 4τ decay time. The parameter ω_d is called the *damped natural frequency*. It is the frequency at which (5) oscillates as it decays. The parameter ω_d is called the *undamped natural frequency*. The response will oscillate at this frequency only if $\zeta = 0$ (i.e. the system is, in fact, undamped). In this case, however, the oscillations will never die out. Such a system is often said to be *neutrally* or *marginally* stable.
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***A 'mantra' for* AERE331:** The characteristic polynomial for an underdamped second order system can always be written as $p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$.

***Example* 1.** Consider the system described by the following differential equation:

$$\ddot{y} + 0.1\dot{y} + 25y = f(t) \quad ; y(0) = y_o \ \dot{y}(0) = v_o$$

(a) Give the system transfer function.

<u>*Answer*</u>: $\frac{Y(s)}{F(s)} \overset{\Delta}{=} G_p(s) = \frac{1}{s^2 + 0.1s + 25}$

(b) Without actually computing the system poles [i.e. the roots of the polynomial $p(s) = s^2 + 0.1s + 25$] determine whether the system is overdamped or underdamped:

<u>*Answer*</u>: Recall that, from the *quadratic formula*, the roots will be complex, if "$b^2 < 4ac$". In relation to $p(s)$ we have $0.1^2 = 0.01 \quad << \quad 4(1)(25) = 100$. Hence, the poles are complex, and hence the system is <u>underdamped</u>.

(c) You should have found that the system poles are complex. Rather than using the quadratic formula to compute them, recall (see the above 'mantra') that for a second order underdamped system, $p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$. Use the method of equating coefficients to find the undamped natural frequency, ω_n, and then the damping ratio, ζ

<u>*Solution*</u>: $p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 0.2s + 25 \quad \Rightarrow \quad \omega_n = 5 \quad \Rightarrow \quad 2\zeta\omega_n = 2\zeta(5) = 0.1 \quad \Rightarrow \quad \zeta = 0.01$.

(d) Compute the system damped natural frequency, time constant, settling time, percent overshoot, rise time, and static gain. [*For the following quantities, see the discussion in the book*.]

<u>*Solution*</u>: $\omega_d \overset{\Delta}{=} \omega_n\sqrt{1-\zeta^2} = 5\sqrt{1-10^{-4}} \cong 5$ rad/sec ; $\tau \overset{\Delta}{=} \frac{1}{\zeta\omega_n} = \frac{1}{0.01(5)} = 20$ sec.

The 4-τ settling time is $t_s = 80$ sec ; $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \cong e^{-\pi\zeta} = e^{0.01\pi} \cong 0.97$ or 97%.

The rise time is $t_r \cong \frac{1.8}{\omega_n} = \frac{1.8}{5} \cong 0.36$ sec. The static gain is $G_p(s=0) = 1/25 = 0.04$

(e) Use the Matlab commands 'tf' and 'step' to arrive at a plot of the unit step response. Then verify the information in part (d).
Solution: To obtain the step response using Matlab, we first define the system transfer function:

```
>> Gp=tf(1,[1 .1 25])
 >>Transfer function:
      1
----------------
s^2 + 0.1 s + 25
```

To get the unit step response for this system, we type
>> step(Gp)

You should be able to use the data cursor to verify the information in (d). [Verified in class.]

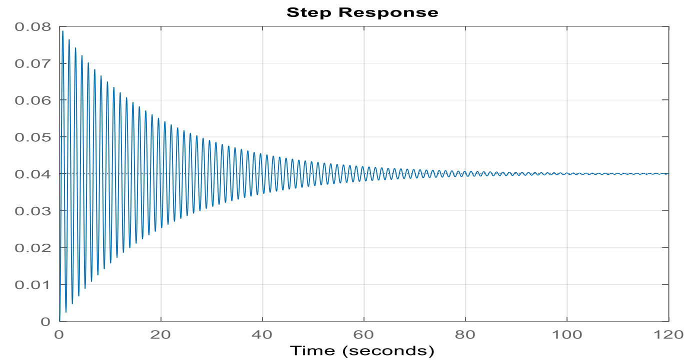


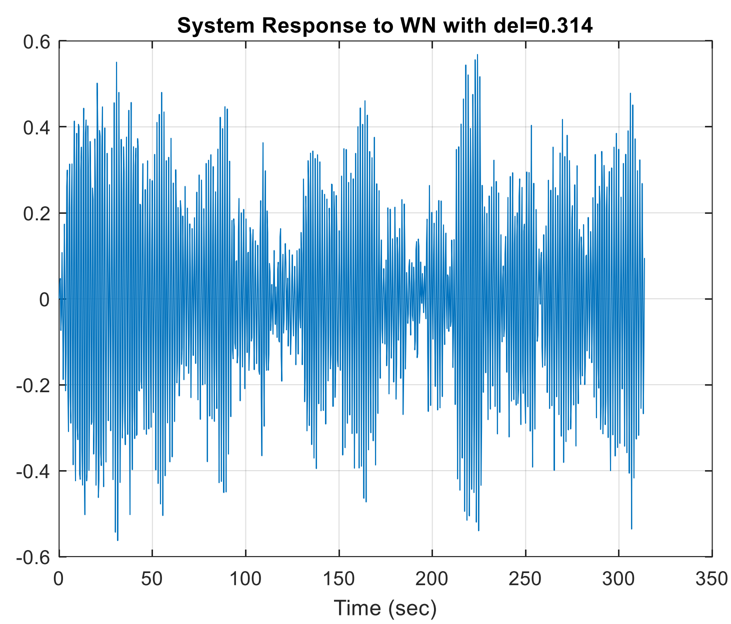
Figure 1. Plant unit step response.

(f) Use the 'lsim' command to obtain a plot of the response of the *continuous-time* (i.e. *analog*) system for an input $\{w(k\Delta)\}_{k=1}^{1000}$ for $\Delta = 0.0314$ sec. Here, the input is what is called a *white noise random process*. Such processes are used as fictitious inputs to a fictitious system whose output is a simulation of a real world process (e.g. turbulence).
Solution: Typing 'help lsim' in the Matlab command window gives:

lsim(SYS,U,T) plots the time response of the dynamic system SYS to the input signal described by U and T. The time vector T is expressed in the time units of SYS and consists of regularly spaced time samples. The matrix U has as many columns as inputs in SYS and its i-th row specifies the input value at time T(i). For example, t = 0:0.01:5;   u = sin(t); lsim(sys,u,t) simulates the response of a single-input model SYS to the input u(t)=sin(t) during 5 time units.

Hence, the needed commands are:

```
%PROGRAM  NAME: tfs573.m    9/27/17
G=tf(1,[1 0.1 25]);
n=1000; del = 0.314;
t=0:n-1; t=t*del;
w=normrnd(0,1,1,n);
y=lsim(G,w,t);
figure(1)
plot(t,y)
title('System Response to WN with del=0.314')
xlabel('Time (sec)')
grid
```



(g) Obtain a plot of the system initial condition response to

Solution: Typing 'help initial' gives: initial(SYS,X0) plots the undriven response of the state-space model SYS (created with SS) with initial condition X0 on the states. This response is characterized by the equations:
  x = A x , y = C x , x(0) = x0

Ouch! We don't have a ss model. We have a tf model. Should we get a ss model from the tf model? But it's Friday ☺