APPENDIX C

Mathematical Review of Laplace Transforms and Matrix Algebra

REVIEW OF MATHEMATICAL CONCEPTS

Laplace Transformation

The Laplace transform is a mathematical technique that has been used extensively in control system synthesis. It is a very powerful mathematical tool for solving differential equations. When the Laplace transformation technique is applied to a differential equation it transforms the differential equation to an algebraic equation. The transformed algebraic equation can be solved for the quantity of interest and then inverted back into the time domain to provide the solution to the differential equation.

The Laplace transformation is a mathematical operation defined by

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt = F(s) \tag{C.1}$$

where $f(t)$ is a function of time. The operator $\mathcal{L}$ and the complex variable s are the Laplace operator and variable, respectively, and $F(s)$ is the transform of $f(t)$. The Laplace transformation of various functions $f(t)$ can be obtained by evaluating Equation (C.1). The process of obtaining $f(t)$ from the Laplace transform $F(s)$, called the inverse Laplace transformation, is given by

$$f(t) = \mathcal{L}^{-1}[F(s)] \tag{C.2}$$

where the inverse Laplace transformation is given by the following integral relationship:

$$f(t) = \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} F(s)\, e^{st}\, ds \tag{C.3}$$

Several examples of Laplace transformations follow.

EXAMPLE PROBLEM C.1. Consider the function $f(t) = e^{-at}$.

Solution. The Laplace transform of this expression yields

$$\mathcal{L}[f(t)] = \mathcal{L}[e^{-at}] = \int_0^\infty e^{-at}\, e^{st}\, dt = \int_0^\infty e^{-(a+s)t}\, dt$$

and the evaluation of the integral gives the transform $F(s)$:

$$F(s) = -\frac{e^{-(a+s)t}}{a+s}\Bigg|_0^{\infty} = \frac{1}{s+a}$$

As another example suppose that $f(t) = \sin \omega t$. Substituting into the definition of the Laplace transformation one obtains

$$F(s) = \mathscr{L}[\sin \omega t] = \int_0^{\infty} \sin \omega t \, e^{-st} \, dt = \frac{1}{2i}\int_0^{\infty} (e^{i\omega t} - e^{i\omega t})\, e^{-st}\, dt$$

Evaluating this integral yields

$$F(s) = \frac{\omega}{s^2 + \omega^2}$$

EXAMPLE PROBLEM C.2. Consider the Laplace transformation of operations such as the derivative and definite integral. When $f(t)$ is a derivative, for example $f(t) = dy/dt$,

$$\mathscr{L}[f(t)] = \int_0^{\infty} \frac{dy}{dt} e^{-st}\, dt$$

Solution. Solution of this integral can be obtained by applying the method of integration by parts. Mathematically integration by parts is given by the following expression:

$$\int_a^b u\, dv = uv\Bigg|_a^b - \int_a^b v\, du$$

Letting u and dv be as follows

$$u = e^{-st}$$

$$dv = \frac{dy}{dt}\, dt$$

then

$$du = -s\, e^{-st}\, dt$$

$$v = y(t)$$

Substituting and integrating by parts yields

$$\mathscr{L}[f(t)] = y(t)\, e^{-st}\Bigg|_0^{\infty} + s\int_0^{\infty} y(t)\, e^{-st}\, dt$$

but the integral

$$\int_0^{\infty} y(t)\, e^{-st}\, dt = Y(s)$$

therefore

$$\mathscr{L}\left[\frac{dy}{dt}\right] = -y(0) + sY(s)$$

In a similar manner the Laplace transformation of higher-order derivatives can be shown to be

$$\mathscr{L}\left[\frac{d^n y}{dt^n}\right] = s^n Y(s) - s^{n-1} y(0) - s^{n-2}\frac{dy}{dt}\Bigg|_{t=0} - \cdots - \frac{d^{n-1}y}{dt^{n-1}}\Bigg|_{t=0}$$

When all initial conditions are 0 the transform simplifies to the following expression.

$$\mathcal{L}\left[\frac{d^n y}{dt^n}\right] = s^n Y(s)$$

Now consider the Laplace transform of a definite integral:

$$\mathcal{L}\left[\int_0^t y(\tau)\, d\tau\right] = \int_0^\infty e^{-st}\, dt \int_0^t y(\tau)\, d\tau$$

This integral can also be evaluated by the method of integration by parts. Letting u and dv be as follows,

$$u = \int_0^t y(\tau)\, d\tau$$

$$dv = e^{-st}\, dt$$

then

$$du = y(t)$$

$$v = \frac{1}{s} e^{-st}$$

Substituting and integrating by parts yields

$$\mathcal{L}\left[\int_0^t y(\tau)\, d\tau\right] = \frac{1}{s} e^{-st} \int_0^t y(\tau)\, d\tau \Big|_0^\infty - \frac{1}{s}\int_0^\infty e^{-st} y(t)\, dt$$

or

$$\mathcal{L}\left[\int_0^t y(\tau)\, d\tau\right] = \frac{Y(s)}{s}$$

By applying the Laplace transformation to various functions of $f(t)$ one can develop a table of transform pairs as shown in Table C.1. This table is a list of some of the most commonly used transform pairs that occur in control system analysis.

TABLE C.1
Table of Laplace transform pairs

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$u(t)$	$1/s$	$\sin \omega t$	$\omega/(s^2 + \omega^2)$
t	$1/s^2$	$\cos \omega t$	$s/(s^2 + \omega^2)$
t^n	$n!/s^{n+1}$	$\sinh \omega t$	$\dfrac{\omega}{s^2 - \omega^2}$
$\delta(t)$ Unit impulse	1	$\cosh \omega t$	$\dfrac{s}{s^2 - \omega^2}$
$\int_{-\varepsilon}^{+\varepsilon} \delta(t)\, dt = 1$		$e^{-at} \sin \omega t$	$\dfrac{\omega}{(s - a)^2 + \omega^2}$
e^{-at}	$1/(s + a)$	$t \cos \omega t$	$\dfrac{s^2 - \omega^2}{(s^2 + \omega^2)}$
$t e^{-at}$	$\dfrac{1}{(s + a)^2}$	$t \sin \omega t$	$\dfrac{2\omega s}{(s^2 + \omega^2)^2}$
$t^n e^{-at}$	$n!/(s + a)^{n+1}$		

Solution of Ordinary Linear Differential Equations

In control system design, a linear differential equation of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t) \tag{C.4}$$

is common. This is a nonhomogeneous linear differential equation with constant coefficients. The Laplace transformations of a differential equation results in an algebraic equation in terms of the transform of the derivatives and the Laplace variables. The resulting algebraic equation can be manipulated to solve for the unknown function $Y(s)$. The expression for $Y(s)$ then can be inverted back into the time domain to determine the solution $y(t)$.

EXAMPLE PROBLEM C.3. Given a second-order differential equation

$$\frac{d^2 y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = \omega_n^2 u(t)$$

where $u(t)$ is a unit step function. Find the solution $y(t)$ if the initial conditions are as follows

$$y(0) = 0$$

$$\frac{dy(0)}{dt} = 0$$

Solution. Taking the Laplace transformation of the differential equation yields

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)Y(s) = \frac{\omega_n^2}{s}$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Now $y(t)$ can be obtained by inverting $Y(s)$ back into the time domain:

$$y(t) = 1 + \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2}\, t - \phi)$$

where $$\phi = \tan^{-1}(\sqrt{1 - \zeta^2} / - \zeta)$$

Partial Fractions Technique for Finding Inverse Transformations

When solving a differential equation using the Laplace transformation approach, the major difficulty is in inverting the transformation back into the time domain. The dependent variable is found as a rational function of the ratio of two polynomials in the Laplace variable, s. The inverse of this function can be obtained by the inverse Laplace transform defined by Equation (C.3). However, in practice it generally is not necessary to evaluate the inverse in this manner. If this function can be found in a table of Laplace transform pairs the solution in the time domain is easily obtained. On the other hand, if the transform cannot be found in the table then an alternate approach must be used. The method of partial fractions reduces the rational fraction to a sum of elementary terms which are available in the Laplace tables.

The Laplace transform of a differential equation typically takes the form of a ratio of polynomials in the Laplace variable, s:

$$F(s) = \frac{N(s)}{D(s)}$$

The denominator can be factored as follows:

$$D(s) = (s + p_1)(s + p_2) \cdots (s + p_n)$$

These roots can be either real or complex conjugate pairs and can be of multiple order. When the roots are real and of order 1 the Laplace transform can be expanded in the following manner:

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p_1)(s + p_2) \cdots (s + p_n)}$$

$$= \frac{C_{p_1}}{s + p_1} + \frac{C_{p_2}}{s + p_2} + \cdots + \frac{C_{p_n}}{s + p_n}$$

where the constants C_{p_i} are defined as

$$C_{p_1} = \left[(s + p_1)\frac{N(s)}{D(s)}\right]_{s=-p_1}$$

$$C_{p_2} = \left[(s + p_2)\frac{N(s)}{D(s)}\right]_{s=-p_2}$$

$$C_{p_i} = \left[(s + p_i)\frac{N(s)}{D(s)}\right]_{s=-p_i}$$

When some of the roots are repeated the Laplace transform can be represented as

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p_1)(s + p_2) \cdots (s + p_i)^r(s + p_n)}$$

and in expanded form as

$$F(s) = \frac{C_{p_1}}{s + p_1} + \frac{C_{p_2}}{s + p_2} + \cdots + \frac{k_1}{(s + p_i)} + \frac{k_2}{(s + p_i)^2} + \cdots + \frac{k_r}{(s + p_i)^r}$$

The coefficients for the nonrepeated roots are determined as shown previously, and the coefficients for the repeated roots can be obtained from the following expression:

$$k_j = \frac{1}{(r - j)!}\frac{\mathrm{d}^{r-j}}{\mathrm{d}s^{r-j}}\left[(s + p_i)^r\frac{N(s)}{D(s)}\right]_{s=-p_i}$$

With the partial fraction technique the Laplace transform of the differential equation can be expressed as a sum of elementary transforms that easily can be inverted to the time domain.

Matrix Algebra

In this section we review some of the properties of matrices. A matrix is a collection of numbers arranged in a square or rectangular array. Matrices are used in the solution of simultaneous equations and are of great utility as a shorthand notation for large systems of equations. A brief review of some of the basic algebraic properties of matrices are presented in the following section.

A rectangular matrix is a collection of elements that can be arranged in rows and columns as follows:

$$\mathbf{A} = a_{ij} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & & & \\ a_{31} & & & \\ a_{41} & & & \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} \end{bmatrix}$$

where the indexes i and j represent the row and column, respectively. The rectangular matrix reduces to a square matrix when $i = j$.

A unit matrix or identity matrix is a square matrix with the elements along the diagonal being unity and all other elements of the array zero. The identity matrix is denoted in the following manner:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Addition and Subtraction of Matrices

Two matrices are equal if they are of the same order; that is, they have the same number of rows and columns and the corresponding elements of the matrices are identical. Mathematically this can be stated as

$$\mathbf{A} = \mathbf{B}$$

if

$$a_{ij} = b_{ij}$$

Matrices can be added provided they are of the same order. Matrix addition is accomplished by adding together corresponding elements.

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

or

$$c_{ij} = a_{ij} + b_{ij}$$

Subtraction of matrices is defined in a similar manner:

$$\mathbf{C} = \mathbf{A} - \mathbf{B}$$

or

$$c_{ij} = a_{ij} - b_{ij}$$

Multiplication of Two Matrices

Two matrices **A** and **B** can be multiplied provided that the number of columns of **A** is equal to the number of rows of **B**. For example, suppose the matrices **A** and **B** are defined as follows:

$$\mathbf{A} = [a_{ij}]_{n,p}$$

$$\mathbf{B} = [b_{ij}]_{q,m}$$

These matrices can be multiplied if the number of columns of **A** is equal to the number of rows of **B**; that is, $p = q$:

$$\mathbf{C} = \mathbf{AB} = [a_{ij}]_{n,p}[b_{ij}]_{q,m} = [c_{ij}]_{n,m}$$

where

$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj} \qquad \begin{array}{l} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m \end{array}$$

EXAMPLE PROBLEM C.4. Given the matrices **A** and **B**, determine the product **AB**:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Solution. **A** and **B** can be multiplied together because the number of columns of **A** is equal to the number of rows of **B**:

$$\mathbf{C} = \mathbf{AB}$$

$$\mathbf{C} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}) \\ (a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}) & (a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}) \end{bmatrix}$$

Some additional properties of matrix multiplication are included in Table C.2. Notice that in general matrix multiplication is not commutative. Multiplication of a

TABLE C.2
Properties of matrix multiplication

$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$	Associative
$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$	Distributive
$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$	Distributive
$\mathbf{AB} \neq \mathbf{BA}$	Commutative

matrix **A** by a scalar constant k is equivalent to multiplying each element of the matrix by the scalar k:

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

Matrix Division (Inverse of a Matrix)

The solution of a system of algebraic equations requires matrix inversion. For example, if a set of algebraic equations can be written in matrix form as

$$\mathbf{Ax} = \mathbf{y}$$

then the solution is given as

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$$

where $\mathbf{A}^{-1}$ is the inverse of the matrix **A**. For the inverse of **A** to exist matrix **A** must be square and nonsingular. The condition that **A** be nonsingular means that the determinant of **A** must be a nonzero value. The inverse of a matrix is defined as follows:

$$\mathbf{A}^{-1} = \frac{\mathrm{Adj}\mathbf{A}}{|\mathbf{A}|}$$

where Adj **A** is called the adjoint of **A**. The adjoint of a matrix is obtained by taking the transpose of the cofactors of the **A** matrix, where the cofactors are determined as follows:

$$C_{ij} = (-1)^{i+j} D_{ij}$$

and D_{ij} is the determinant obtained by eliminating the ith row and jth column of **A**. Some additional properties of the inverse matrix are given in Table C.3.

The transpose of a matrix is obtained by interchanging the rows and columns of the matrix. Given the matrix **A**,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

TABLE C.3
Properties of an inverse matrix

1. $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
2. $[\mathbf{A}^{-1}]^{-1} = \mathbf{A}$
3. If **A** and **B** are nonsingular and square matrices then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

then the transpose of **A** is

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

For additional properties of matrices the reader should consult his or her mathematics library.