Lecture 12/4-6/19

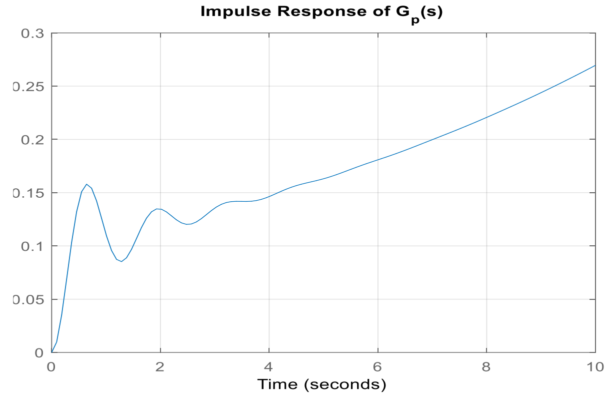

Figure 1 Plot of $g_p(t)$.

In this lecture we investigated the situation where the plane was statically unstable, but had a stable Dutch roll mode. The transfer function for this situation was chosen to be:

$$\frac{\psi(s)}{\delta_r(s)} = \frac{25}{(10s-1)(s^2+2s+25)} \stackrel{\Delta}{=} G_p(s). \qquad (1)$$

The impulse response $g_p(t)$ is shown at right. Clearly, the system is unstable.

Next, we considered the use of unity feedback proportional control to stabilize (1). The block diagram is shown at right. In class it was shown that the closed loop transfer function associated with Figure 2 is:

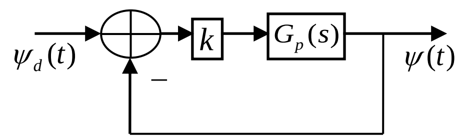

Figure 2 Closed loop block diagram.

$$W(s) \stackrel{\Delta}{=} \frac{\psi(s)}{\psi_d(s)} = \frac{25K}{(10s-1)(s^2+2s+25)+25K} = \frac{25K}{10s^3+19s^2+248s+25(K-1)}. \qquad (2)$$

The system (2) will be stable only if all three roots of $p(s) = 10s^3+19s^2+248s+25(K-1)$ are in the proper Left Half Plane (LHP). To find the range of stabilizing *K*-values, one could use the Routh array:

s^3 :	10	248	0
s^2 :	19	$25(K-1)$	0
s^1 :	c	0	0
s^0 :	$25(K-1)$	0	0

, where $c = \frac{19(248)-250(K-1)}{19}$.

The first column of the array will have no sign changes for the *K*-values $1 < K < 19.848$. Hence, this is the range of stabilizing *K*-values.

However, the fact that (2) is stable for $1 < K < 19.848$ does not mean that it is desirable. Consider the impulse response shown at right for $K = 15$. The response is stable, but totally unacceptable.

The use of the Routh array, as valuable as it is, is an exercise in algebra. To gain a more visual appreciation, we will now address the use of the root locus; which is a geometric approach.

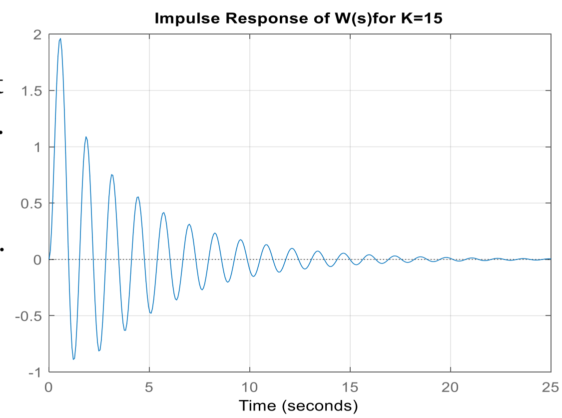

Figure 3 Closed loop impulse response for $K = 15$.

We are interested in the CL system roots that are the solution to:

$$p(s) = 10s^3+19s^2+248s-25+25K = 0. \qquad (3)$$

For any chosen value of *K*, (3) will have three roots. Hence, as we vary *K*, there will be three trajectories of roots in the complex plane. These trajectories are also called loci; hence, the term ***root locus***.

Now, to arrive at the root locus associated with (3), we could simply use the 'roots' command, as you have done in past homeworks. However, this is what I would call a 'turn the crank' method. It lends little geometric insight as to what's really going on. We will now approach the problem of constructing the root locus for (1) in a less rote manner. More importantly, this approach will lead one to naturally posit control structures that can improve matters.

To this end, note that (3) is equivalent to:

$$1+\frac{25K}{10s^3+19s^2+248s-25}=0. \quad (4)$$

The only caveat in re-casting (3) as (4) is that (4) is only well-defined for *s*-values that are not the roots of $p_{OL}(s)=10s^3+19s^2+248s-25$. For any one of those three roots, (3) will be zero if and only if $K=0$.

Now, recall that $p_{OL}(s)=10s^3+19s^2+248s-25$ is the denominator polynomial of the plant transfer function (1). However, in view of Figure 2, (1) is also the *open loop transfer function*; i.e. that transfer function one arrives at by opening the feedback loop at the summing junction, and then gathering up all the transfer functions in that loop. Hence, the subscript *OL* in $p_{OL}(s)$. In gathering up the transfer functions, we arrive at the open loop transfer function:

$$G_{OL}(s)=\frac{25K}{10s^3+19s^2+248s-25}\overset{\Delta}{=}KG(s). \quad (5)$$

From (5), we see that in this example (4) becomes:

$$1+KG(s)=0. \quad (6)$$

This leads us to the Matlab command 'rlocus'. While it is natural to assume that the required argument for this command would be the polynomial (1), whose root locus is desired. However, generally, such a polynomial could problematic to express as a function of *K*. On the other hand, (4), which is equivalent to (3), is not at all problematic. We simply give $G(s)$ as the argument of 'rlocus'. The commands below resulted in the root locus plot in Figure 4:

```
G=tf(25,[10 19 248 -25]);   rlocus(G)  grid
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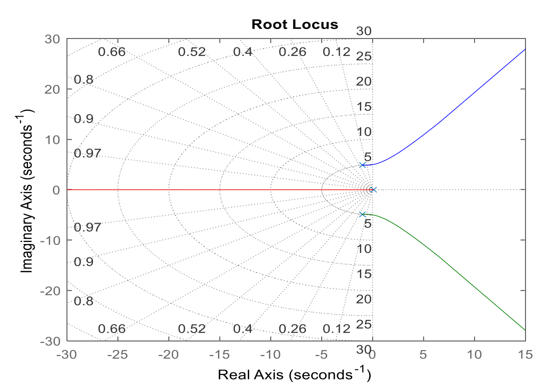

Figure 4 Root locus associated with (3).

The x's in Figure 4 are the roots of (3) when $K=0$. These roots are exactly the poles of the open loop transfer function. As *K* is increased, the roots of (3) depart from the OL poles in the manner shown in Figure 4. Specifically, the real root locus begins at $s_1=1+i0$ and moves toward $s_1=-\infty+i0$ as *K* is increased. The other two roots that are a complex conjugate pair $s_{2,3}=-0.94\pm i4.89$ for $K=0$ retain their complex conjugate structure as $K\to\infty$. Moreover, they enter the RHP.

We knew that this would be the case from analysis of the first column of the Routh array, since, for sufficiently large K there are two sign changes in the first column. Furthermore, the zoomed plot of Figure 4 given in Figure 5 verifies the Routh array result that all the roots of (3) will be in the LHP for $1 < K < 19.848$.

Specifically, we see that the real root enters the LHP for $K = 1$, and the conjugate pair of roots enter the RHP for $K = 19.9$.

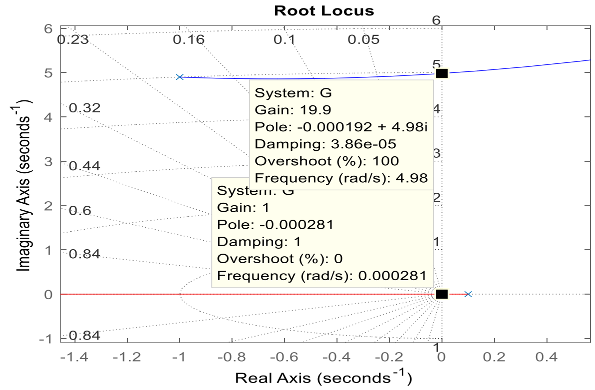

Figure 5 Zoomed plot of Figure 4.

Now, let's go a little deeper into the rabbit hole ☺. Write (6) as:

$$KG(s) = -1 + i0 = 1e^{\pm i\pi} . \tag{7}$$

Furthermore, write the OL transfer function (5) as:

$$G_{OL}(s) = \frac{25K}{10s^3 + 19s^2 + 248s - 25} = \left(\frac{25K}{10}\right)\frac{1}{s^3 + 1.9s^2 + 24.8s - 2.5} = \mathrm{K}\frac{1}{(s-p_1)(s-p_2)(s-p_3)} . \tag{8}$$

To connect (8) to (7), consider the generic term $s - p$. This is the difference between two vectors (i.e. numbers in the complex plane), as is shown at right. We can write this difference vector in polar form as:

$$s - p = l_p e^{i\theta_p} . \tag{9}$$

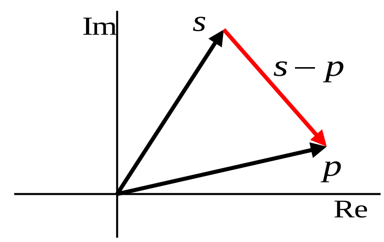

Figure 6 Plot of $s - p$.

From (9), we can write (8) as:

$$G_{OL}(s) = \mathrm{K}\frac{1}{(s-p_1)(s-p_2)(s-p_3)} = \frac{\mathrm{K}}{\prod_{k=1}^{3} l_{p_k}} e^{-i\sum_{k=1}^{3}\theta_{p_k}} . \tag{10}$$

Finally, equating (7) and (10) gives:

$$\frac{\mathrm{K}}{\prod_{k=1}^{3} l_{p_k}} e^{-i\sum_{k=1}^{3}\theta_{p_k}} = 1e^{\pm i\pi} . \tag{11}$$

From (11) we see that two conditions are required for s to be on the root locus:

(C1) The magnitude condition: $\frac{\mathrm{K}}{\prod_{k=1}^{3} l_{p_k}} = 1$ and (C2) The angle condition: $-\sum_{k=1}^{3} \theta_{p_k} = \pm\pi$. (12)

The condition (C1) is trivial, in the sense that for any s, regardless of whether or not it is on the root locus, we can satisfy this condition by setting $\mathrm{K} = \prod_{k=1}^{3} l_{p_k}$. It is (C2) that determines the geometric structure of the root locus.

In-Class 1: Use (C2) to prove that $s = -2 + i2$ is not on the root locus of Figure 4.

In class we computed $-\sum_{k=1}^{3} \theta_{p_k} = -520^o = -180^o - \mathbf{340^o}$. In words, the 'defect angle' associated with $s = -2 + i2$ is -340^o.

In-Class 2: What would need to be added to $G_{OL}(s)$ so that $s = -2 + i2$ is on the root locus?

We could add zeros to $G_{OL}(s)$ such that the angles between them and $s = -2 + i2$ add up to 340^o.