Aircraft Response Due To Atmospheric Inputs (c.f. Nelson Chapter 6 p.212)

We begin this section by repeating the longitudinal and lateral small disturbance dynamical system equations, but with simplifications obtained by neglecting variables that have been described as negligible. Neglecting $M_{\dot{w}}$, the longitudinal state equation on p.12 of the Chapter 4 notes is:

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{w} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_\delta & X_{\delta_T} \\ Z_\delta & Z_{\delta_T} \\ M_\delta & M_{\delta_T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\delta_T \end{bmatrix}. \qquad (1)$$

By setting the product of inertia $I_{xz} = 0$, the lateral equations on p.4.44 of these notes are:

$$\begin{bmatrix} \Delta\dot{v} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & Y_r - u_0 & g\cos\theta_0 \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{bmatrix} + \begin{bmatrix} 0 & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix}. \qquad (2)$$

The atmosphere is not a uniform medium. It includes spatial and temporal dynamics associated with wind. Recall that the forces and moments acting on a plane depend on its motion relative to the local atmosphere. To demonstrate how the above equations are modified due to wind, we can write the plane *x*-axis velocity disturbance in relation to the local *x*-axis ***gust*** velocity, u_g, as: $\Delta u_a = \Delta u - u_g$. Repeating this for all disturbance variables gives:

linear velocities: $\Delta u_a = \Delta u - u_g \quad ; \quad \Delta v_a = \Delta v - v_g \quad ; \quad \Delta w_a = \Delta w - w_g$

rotational velocities: $\Delta p_a = \Delta p - p_g \quad ; \quad \Delta q_a = \Delta q - q_g \quad ; \quad \Delta r_a = \Delta r - r_g$.

The above state equations then become:

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{w} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_\delta & X_{\delta_T} \\ Z_\delta & Z_{\delta_T} \\ M_\delta & M_{\delta_T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\delta_T \end{bmatrix} + \begin{bmatrix} -X_u & -X_w & 0 \\ -Z_u & -Z_w & 0 \\ -M_u & -M_w & -M_q \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_g \\ w_g \\ q_g \end{bmatrix} \qquad (3)$$

$$\begin{bmatrix} \Delta\dot{v} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & Y_r - u_0 & g \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{bmatrix} + \begin{bmatrix} 0 & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix} + \begin{bmatrix} -Y_v & 0 & 0 \\ -L_v & -L_p & -L_r \\ -N_v & -N_p & -N_r \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_g \\ p_g \\ r_g \end{bmatrix}. \qquad (4)$$

Both (3) and (4) have the structure of the following dynamical system:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}_1\mathbf{u}_1 + \mathbf{B}_2\mathbf{u}_2 \,.$$

In Chapter 4 we ignored the control inputs. We will do the same here. In this case, the above form becomes:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \,. \qquad (5)$$

In both (3) and (4) the input, **u**, is a 3-D vector, and the output **x** is a 4-D vector. Hence, in the jargon of modern control theory, these are said to be examples of a *Multi-Input/Multi-Output* (*MIMO*) system.

For a *MIMO*(3,4) system, each of the 3 inputs relates to each of the 4 outputs. Hence, there are a total of 12 scalar-valued relationships. The *transfer function* associated with (5) is obtained by taking its Laplace transform under *zero initial conditions*. Doing this, we obtain $s\mathbf{X}(s) = \mathbf{AX}(s) + \mathbf{BU}(s)$, or:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{BU}(s) \;\overset{\Delta}{=}\; \mathbf{H}(s)\mathbf{U}(s) \,. \qquad (6)$$

The matrix $(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \;\overset{\Delta}{=}\; \mathbf{H}(s)$ defined in (6) is the *transfer function matrix* that relates the input, *u*, to the output *x*.

In-Class Question: What is the dimension of $\mathbf{H}(s)$? **Answer: 4x3**

The (*j*,*k*) element of $\mathbf{H}(s)$ is a scalar-values transfer function that relates the j^{th} output to the k^{th} input. For example, in (1) the transfer function between the input $\Delta\delta(t)$ and the output $\Delta u(t)$ is the (1,1) element of the transfer function matrix.

We are now in a position to compute the transfer function matrices associated with (1) and with (2) . We can do this simply by identifying the specific matrices associated with the matrices **A** and **B** in $(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \mathbf{H}(s)$. To obtain the 12 scalar-valued transfer functions for each system by hand would be a very laborious task. It is conceivable that a symbolic code (e.g. *Mathematica*) could perform this task and yield nice expressions. We will not undertake either task at this time. The motivated student is encouraged to 'go for it'!

A Closer Look at Scalar-Valued Transfer Functions

We have already seen the value of having a scalar-valued transfer function. It allows us to specify an input, *u*, and obtain the output, *x*, using very simple mathematics, a table of Laplace transforms, and *Matlab* commands. To motivate the discussion, we will consider the "PURE VERTICAL OR PLUNGING MOTION" of a plane, considered in Nelson, p.218. In this section a plane constrained to undergo only vertical motion in response to a gust of wind is addressed. The development begins with:

$$\sum Forces\ in\ z-direction \quad = \quad m\,dw/dt \quad = Z + W \,.$$

Expressing the aerodynamic force in the *z*-direction as $Z = Z_0 + \Delta Z$ (i.e. a small perturbation force), results in $Z_0 = -W$ (i.e. equilibrium), and writing *w* as $w \overset{\Delta}{=} \Delta w$ (i.e. unaccelerated flight) gives:

$$\Delta Z / m = d\Delta w / dt \,.$$

Since the vertical force is a function of the angle of attack, $\Delta\alpha$, and its derivative, $\Delta\dot{\alpha}$, this can be expressed as:

$$Z_\alpha \Delta\alpha + Z_{\dot\alpha}\Delta\dot\alpha \quad - \quad \Delta\dot w = \quad 0. \tag{1}$$

The change in the angle of attack due to the combination of its motion and the vertical wind gust is:

$$\Delta\alpha \cong \Delta w / u_0 \quad - \quad w_g / u_0 . \tag{2a}$$

This gives: $$\Delta\dot\alpha \cong \Delta\dot w / u_0 - \dot w_g / u_0 . \tag{2b}$$

$$(1 - Z_{\dot\alpha}/u_0)\Delta\dot w - (Z_\alpha / u_0)\Delta w = \quad -\left[(Z_{\dot\alpha}/u_0)\dot w_g + (Z_\alpha / u_0) w_g\right]. \tag{3}$$

As mentioned before, one standard for of a first order model has the form $\tau \dot y(t) + y(t) = g_s f(t)$, where the 'input' is $f(t)$, the 'output' is $y(t)$, the system *time constant* is τ, and the system *static gain* is g_s. putting (3) in this form gives:

$$-\left(\frac{u_0 - Z_{\dot\alpha}}{Z_\alpha}\right)\Delta\dot w + \Delta w = \quad (Z_{\dot\alpha}/Z_\alpha)\dot w_g + w_g . \tag{4}$$

Hence, the system time constant is $\tau = -\left(\frac{u_0 + Z_{\dot\alpha}}{Z_\alpha}\right)$ and the static gain is $g_s = 1$. The system transfer function is obtained by taking the Laplace transform of (4) assuming zero initial conditions. It is:

$$H(s) \stackrel{\Delta}{=} \frac{\Delta W(s)}{W_g(s)} = \frac{(Z_{\dot\alpha}/Z_\alpha)s + 1}{-\left(\frac{u_0 + Z_{\dot\alpha}}{Z_\alpha}\right)s + 1} = \frac{(Z_{\dot\alpha}/Z_\alpha)s + 1}{\tau s + 1} . \tag{5}$$

From (5), we can easily obtain the response for a variety of gust profiles. We now consider some progressively realistic profiles.

***Case* 1:** A very idealized (and mathematically simple) gust profile.

Suppose that the gust profile, $w_g(t)$ has the form of a short burst of duration, *T*, and strength, W_o. To obtain its Laplace transform, $W_g(s)$, we need three items from a table of Laplace transforms:

<u>Item 1</u>: The *unit step* function, $u_s(t)$ has the form $u_s(t) = 1 \; for \; t > 0.$ Its Laplace transform is $U_s(s) = \frac{1}{s}$.

<u>Item 2</u>: For any function $f(t)$ with Laplace transform $F(s)$, the time-shifted function $r(t) \stackrel{\Delta}{=} f(t - t_o)$ has Laplace transform $R(s) = F(s)e^{-st_o}$.

<u>Item 3</u>: For any function $f(t)$ with Laplace transform $F(s)$, the scaled function $r(t) \stackrel{\Delta}{=} c\, f(t)$ has Laplace transform $R(s) = cF(s)$.

The idealized gust pulse can be written as: $w_g(t) = W_o[u_s(t) - u_s(t - T)]$. Using the above items, its Laplace transform is:

$$W_g(s) = W_o\left(\frac{1-e^{-sT}}{s}\right). \tag{6}$$

From (5) and (6), the Laplace transform of the response $\Delta w(t)$ is:

$$\Delta W(s) = H(s)W_g(s) = \left[\frac{(Z_{\dot{\alpha}}/Z_\alpha)s+1}{\tau s+1}\right]\left(\frac{1-e^{-sT}}{s}\right). \tag{7}$$

We can use a table of transform pairs to obtain the expression for $\Delta w(t)$. But it is instructive to express (7) as the sum of two components:

$$\Delta W(s) = H(s)W_o\left(\frac{1-e^{-sT}}{s}\right) = H(s)W_o\left(\frac{1}{s}\right) \quad - \quad H(s)W_o\left(\frac{1}{s}\right)e^{-sT}. \tag{8}$$

The first term on the right side of (8) is simply the scaled step response of the system $H(s)$. The second term on the right is the same response, but time-delayed by an amount *T*.

EXAMPLE PROBLEM 6.1 of Nelson (p.224) As above, both Nelson and Etkin typically assume that $Z_{\dot{\alpha}} \cong 0$. Then we have $\tau = -u_0 / Z_\alpha$. Recall that $Z_\alpha = -C_{L_\alpha} QS/m$. Hence, $\tau = 2W/(C_{L_\alpha} S\rho u_0 g)$. For a general aviation plane flying at 125 fps, we obtain $\tau \cong 0.7\sec$ With these numerical values, the system transfer function is:

$$H(s) \overset{\Delta}{=} \frac{\Delta W(s)}{W_g(s)} = \frac{1}{0.7s+1} = 1.43\left(\frac{1}{s+1.43}\right).$$

From entry #4 in the table of Laplace transform pairs, the corresponding system impulse response is:

$$h(t) = 1.43e^{-1.43t}.$$

We will assume that the gust magnitude is $W_o = 15\ ft/\sec$, and that its duration is $T = 2\sec.$. From entry #7 of the table and equation (8) above, the system response to this gust is:

$$\Delta w(t) = 15\left[(1-1.43e^{-1.43t}) - (1-1.43e^{-1.43(t-2)})u_s(t-2)\right]. \tag{9}$$

Note that in (9) we cannot cancel the two 1's, since the second 1 is multiplied by the time-delayed unit step function. Consequently for any time $t < 2\sec$, (9) is only:

$$\Delta w(t) = 15(1-1.43e^{-1.43t}) \quad for \quad 0 < t < 2\sec. \tag{10}$$

To plot the response (9) using *Matlab*:

Define an array of discrete time values that extend from 0 to at least $(2+4\tau)\sec.$ We will choose the maximum time to be 5 seconds. The interval, ΔT, between samples should be small relative to τ. We will choose $\Delta T = 0.01\sec.$ Hence the array of computation times will include a total of 1+5/.01=501 numbers. This array is computed as: tvec = 0:500; tvec = tvec*.01. The response portion given by (10) is then simply: w = 15*(1 – exp(-1.43*tvec). Its time-delayed version begins after the 2/.01=200th time index. And so: ws = [zeroes(1,200) w]; ws = ws(1:501).

These commands are summarized as:

```
>> tvec=0:500; tvec=.01*tvec;
>> w=15*(1-exp(-1.43*tvec));
>> ws=[zeros(1,200) w]; ws=ws(1:501);
>> dw=w - ws;
>> plot(tvec,dw)
```

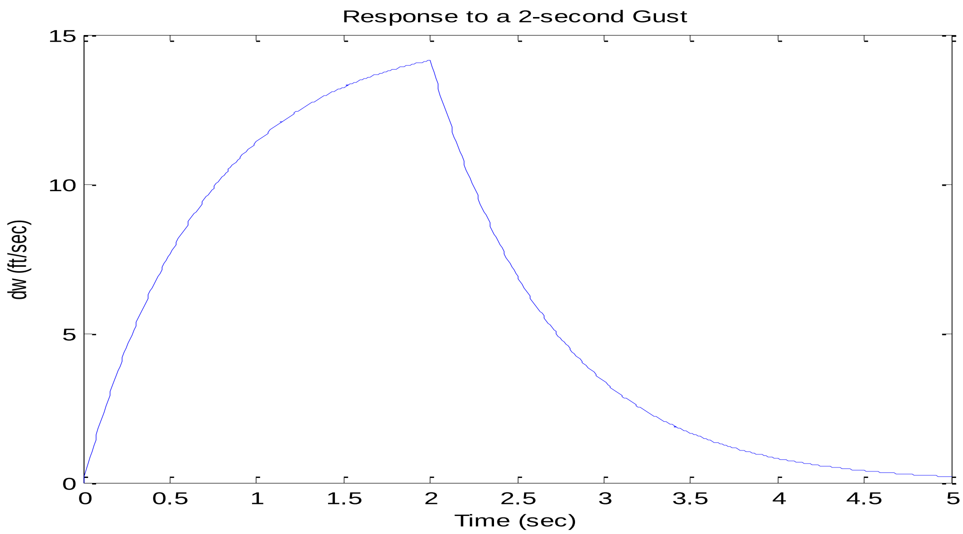

Figure 1. Plane vertical speed response to a 2-second idealized gust with magnitude 15 ft/sec.

FREQUENCY CONTENT ASSOCIATED WITH INPUTS, OUasTPUTS & TRANSFER FUNCTIONS

The profile in *Case* 1 was mathematically convenient, and does provide some basic insight as to how a gust might affect the dynamics of an aircraft. But anyone who has stood in an open field on a windy day knows that gusts of wind, and the wind, in general, are more complicated.

If you watch a tree during a gust, you will find that it doesn't simply bend in a static way. Rather, it sways. That's because a tree is an underdamped system with a natural frequency. If you give it an initial condition by pulling it with a rope, and then let go, it will sway back and forth at its damped natural frequency. Now, the only way that a natural frequency can be excited is if the input has energy at that frequency. And so, returning to models for a wind gust, if the chosen model has no energy at the natural frequency of the plane dynamics, then the response to that gust model will not involve those dynamics. This could result in very misleading predictions to the plane's response to a real gust.

The key word that was used repeatedly here is ***frequency***. To this point, we have been concerned with the relation between a transfer function and *time domain* behavior (e.g. settling time and period of oscillation). And so, a natural question is:

QUESTION: *How does a system transfer function, say,* $H(s)$ *, convey frequency information about the system dynamics?*

To begin to answer this question, we will now prove a most fundamentally important connection between the *time-domain* (*t*) and *Laplace-domain* (*s*) representations of a system with transfer function $H(s)$. To emphasize this relationship, we will not state it as a fact. Rather, we will state it as a theorem. The proof of this theorem is central to understanding it.

THEOREM. A system transfer function, say, $H(s)$, is mathematically identical to the Laplace transform of the system *impulse response.*

Before proving this theorem, let's be clear about the difference between a transfer function, $H(s)$, and the Laplace domain expression for the system response to an input, say, $F(s)$. Suppose, for example, that the input is a force, with units of lb_f and that the response is displacement, with units of *ft*. Then the units of $H(s)$ are [ft / lb_f], while the units of the response, $Y(s) = H(s)F(s)$ are *ft*. And so to say that the Laplace transform of the response to a unit impulse, $\delta(t)$ is equal to $H(s)$ makes no sense. They have different units! The above theorem does not say they are equal. T says they have the same mathematical expression. We will use the force/displacement setting in the proof.

PROOF: For a unit impulse input, $f(t) = \delta(t)$, the Laplace transform is $F(s) = 1\,[lb_f]$. The Laplace transform of the response, $y(t)$, to <u>any</u> input, $f(t)$, is simply $Y(s) = H(s)F(s)\,[ft]$. Hence, for the unit impulse input, this becomes $Y(s) = H(s)\,[ft]$. Recall, that the standard convention is to use a lower case for the time domain expression, and the upper case for its Laplace transform. And so, since $Y(s) = H(s)\,[ft]$ we have $y(t) = h(t)\,[ft]$. The time-domain function $h(t)$ is the inverse Laplace transform of the system transfer function, $H(s)$. It is also the time-domain response to a unit-impulse input. Hence, $h(t)$ is called the system *impulse response function.* In summary then, ***the system transfer function is the Laplace transform of the system impulse response function.*** □

EXAMPLE PROBLEM 6.1 *continued*: The impulse response of the system with transfer function $H(s) \overset{\Delta}{=} \dfrac{\Delta W(s)}{W_g(s)} = \dfrac{1}{0.7s+1} = 1.43\left(\dfrac{1}{s+1.43}\right)$ is $h(t) = 1.43e^{-1.43t}$.

This response is plotted against the response to a 2-second wind gust below.

Figure 2. Plane vertical speed response to a 2-second idealized gust with magnitude 15 ft/sec, and to an impulse with intensity equal to 30.

Discussion: The response to the 2-second pulse is well-behaved at $t = 0$, whereas the impulse response makes a very sharp jump. This sharp jump means that the frequency content of the impulse response will include much higher frequencies that the frequency content in the 2-second pulse. But this may not be at all obvious to those who are new to this topic. To quantify this, we now give the answer to the above QUESTION:

ANSWER: The frequency content associated with a system with transfer function $H(s)$ is obtained by setting $s = i\omega$. The quantity $H(i\omega)$ is called the system ***Frequency Response Function*** (FRF).

It follows that the frequency content associated with the response $y(t)$, to <u>any</u> input, $f(t)$, is simply $Y(i\omega) = H(i\omega)F(i\omega)$. For completeness, we give the following definition.

Definition 1. Let $f(t)$ be a function of time, and assume that $f(t) = 0$ *for* $t < 0$. The function $F(s) = \int_{t=0}^{\infty} f(t)e^{-st}dt$ is called the ***Laplace Transform*** of $f(t)$. For $s = i\omega$, this transform becomes $F(i\omega) = \int_{t=0}^{\infty} f(t)e^{-i\omega t}dt$, and is called the ***Fourier Transform*** of $f(t)$.

We are now in a position to quantitatively describe the frequency content associated with the two responses in Figure 2 above.

The frequency content associated with the response to the 2-second gust is obtained directly from (7) (in the context of EXAMPLE PROBLEM 6.1) as:

$$\Delta W_{2\sec}(i\omega) = H(i\omega)W_{g-2\sec}(i\omega) = \left[\frac{1.43}{1.43+i\omega}\right]\left(\frac{1-e^{-i2\omega}}{i\omega}\right), \tag{10a}$$

whereas, the frequency content associated with the impulse gust (with intensity = 30) is simply:

$$\Delta W_{impulse}(i\omega) = H(i\omega)W_{g-impulse}(i\omega) = \left[\frac{1.43}{1.43+i\omega}\right]30\,. \tag{10b}$$

The frequency content associated with (10a) and (10b) is shown below.

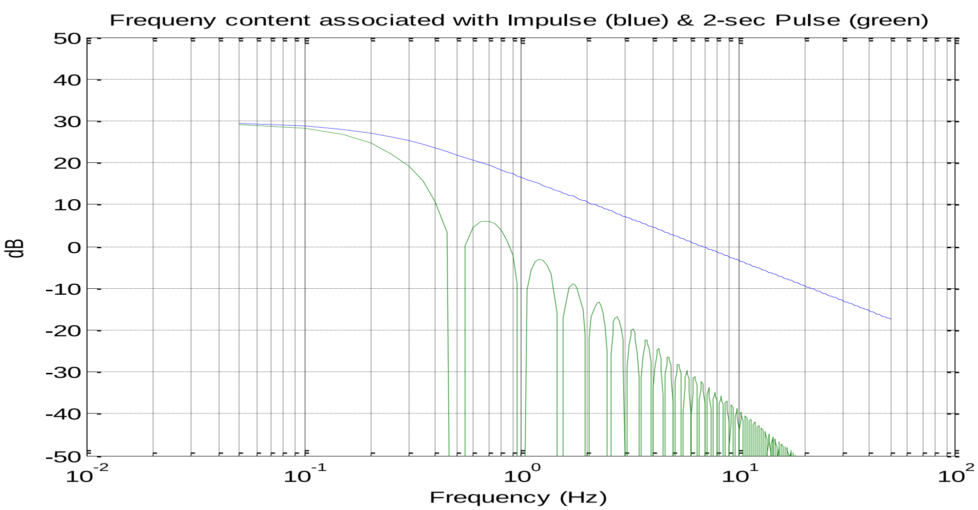

Figure 3. The frequency content associated with (10a) BLUE and (10b) GREEN. NOTE: (10b) is the system (scaled) FRF.

Discussion: The frequency content of the two inputs is similar up to about 0.1 Hz. At higher frequencies the frequency content of the response to the 2-second pulse drops significantly relative to that associated with the impulse.

THE FREQUENCY CONTENT IN A RANDOM WIND PROFILE

Again, imagine you are standing in a field during gusty wind conditions. Clearly, no two gusts will feel exactly the same. In fact, the wind, in general is not exactly the same from second to second. And so, a more realistic model for a gust would be one that has elements of randomness to it. At issue here is not the energy content in a gust. Every gust will have different energy content. What is of central concern now is the *expected* frequency content. More specifically, it is the expected energy at each frequency.

Consider a function $f(t)$. Its frequency description is given by its Fourier Transform:

$F(i\omega) = \int_0^\infty f(t)e^{-i\omega t}dt$. This is well-defined if $f(t)$ decays fast enough as $t \to \infty$. (e.g. a decaying exponential.) But wind (esp. upper atmospheric wind) is <u>*always*</u> present. It does not simply go away after some 4τ period of time. Furthermore, it has randomness. In fact, wind is an example of a <u>*random process*</u>. In this section we will assume it is a random process that has the same general behavior at in time. We will refer to such a process as a (weakly) **stationary random process.**

Definition 2. Assume that $f(t)$ is a stationary random process. Then the Fourier Transform of $f(t)$ is defined as

$$F(i\omega) \stackrel{\Delta}{=} \lim_{T\to\infty} \frac{1}{\sqrt{T}} \int_0^T f(t)e^{-i\omega t}dt \;\; for\; 0 \le \omega < \infty.$$

Remark: The Fourier Transform operation converts a time-domain random function, $f(t)$; $0 \le t < \infty$, to a random function $F(i\omega)$; $0 \le \omega < \infty$. It is simply a different way of viewing the random function (i.e. as a function of ω instead of as a function of *t*). It does not reduce the level of variability.

Since $F(i\omega)$; $0 \le \omega < \infty$ is a random function, we need to address its frequency content in relation to the ***expected energy at any frequency***. This is defined as:

Definition 3. The ***expected energy*** at a frequency ω is defined as $E[|F(i\omega)|^2]$. In Nelson (p.227-228) the symbol used for it is $E[|F(i\omega)|^2] \stackrel{\Delta}{=} \Phi_f(\omega)$. In words, $\Phi_f(\omega)$ is called the ***power spectral density*** (*psd*) function associated with the random process $f(t)$. The symbol $E(\;)$ denotes the *expected value* operation.

Examples of Wind *PSD* Models in Relation to the (*u,v,w*) directions:

For a specified plane speed, u_0, define the scaled frequency $\Omega \stackrel{\Delta}{=} \omega / u_0$. The following are the von Karman *psd* models for the three directions (*u,v,w*):

$$\Phi_{u_g}(\Omega)=\sigma_{u_g}^2\frac{2L_{u_g}}{\pi}\frac{1}{[1+(1.339L_{u_g}\Omega^2]^{5/6}} \tag{11a}$$

$$\Phi_{v_g}(\Omega)=\sigma_{v_g}^2\frac{2L_{v_g}}{\pi}\frac{1}{[1+(1.339L_{v_g}\Omega^2]^{11/6}} \tag{11b}$$

$$\Phi_{w_g}(\Omega)=\sigma_{w_g}^2\frac{2L_{w_g}}{\pi}\frac{1}{[1+(1.339L_{w_g}\Omega^2]^{11/6}} \tag{11c}$$

The model parameter σ^2 controls the overall intensity of the *psd* for a given direction, and the parameter L controls the turbulence scale in a given direction. These parameters are specified by the researcher.

EXAMPLE 6.1 *continued*: Since we are concerned in this example with vertical gusts, we will use equation (11c), but will express it as a function of $\omega=\Omega u_0$:

$$\Phi_{w_g}(\omega)=\frac{\left(\sigma_{w_g}^2\frac{2L_{w_g}}{\pi}\right)}{\left[1+\left(\frac{1.339L_{w_g}}{u_0^2}\right)\omega^2\right]^{11/6}}$$

<u>Connection to *Matlab Simulink* & to Military Specifications</u>

Von Karman Wind Turbulence Model (Continuous): Generate continuous wind turbulence with Von Kármán velocity spectra. https://www.mathworks.com/help/aeroblks/vonkarmanwindturbulencemodelcontinuous.html

The Dryden model is found in: https://www.mathworks.com/help/aeroblks/drydenwindturbulencemodelcontinuous.html

Library: Environment/Wind

Description: *The Von Kármán Wind Turbulence Model (Continuous) block uses the Von Kármán spectral representation to add turbulence to the aerospace model by passing band-limited white noise through appropriate forming filters. This block implements the mathematical representation in the* ***Military Specification MIL-F-8785C and Military Handbook MIL-HDBK-1797.***

According to the military references, turbulence is a stochastic process defined by velocity spectra. For an aircraft flying at a speed V through a frozen turbulence field with a spatial frequency of Ω radians per meter, the circular frequency ω is calculated by multiplying V by Ω. The following table displays the component spectra functions: [See Matlab documentation] □

Major Points re: AerE355:

1. In the real world, wind velocity is not modeled as a simple deterministic function.

2. Military specs. require that wind velocity be modeled by *passing band-limited white noise through appropriate forming filters.* .

Q1: What in the world is *band-limited white noise?*

A1: To answer this, we first need to understand what w*hite noise* is. A random process, call it $w(t)$, is a ***white noise*** process if its *psd* is: $\Phi_u(\omega) = \sigma_u^2$ (i.e. it is a flat *psd*.) It follows that band-limited white noise is white noise that has been passed through a *low-pass filter.*

Q2: What is a *low-pass filter?*

A2: It is a transfer function whose FRF magnitude is unity at low frequencies and decays to zero at higher frequencies. The following first order system is an example of a low-pass filter.

Example of a low-pass filter- Consider the first order system with transfer function $H(s) = \dfrac{g_s}{\tau s+1}$. Recall, that the parameter τ is the system *time constant*, and that g_s is the system static gain. The system Frequency Response Function (FRF) is:

$$H(i\omega) = \frac{g_s}{\tau(i\omega)+1} \qquad (12a)$$

Since (12a) is a complex number, we can express it in polar form. [Recall that the polar form of the complex number $a+ib$ is $\sqrt{a^2+b^2}\ e^{i\theta}$ where $\theta = \tan^{-1}(b/a)$ if both $a,b \ge 0$.] Hence,

$$\begin{aligned} &H(i\omega) = \frac{g_s}{\tau(i\omega)+1} = e^{i\theta(\omega)} = M(\omega)e^{i\theta(\omega)} \\ &where\ M(\omega) = \frac{g_s}{\sqrt{1+(\tau\omega)^2}} \quad and \quad \theta(\omega) = -\tan^{-1}(\tau\omega) \end{aligned} \qquad (12b)$$

It is standard practice to express the magnitude, $M(\omega)$, of the FRF in units called *decibels* (dB). Specifically: $M_{dB}(\omega) \overset{\Delta}{=} 20\log_{10}[M(\omega)]$. Hence, in relation to (12), we have:

$$M_{dB}(\omega) = 20\log_{10}\left(\frac{g_s}{\sqrt{1+(\tau\omega)^2}}\right) = 20\log_{10}(g_s) - 10\log_{10}[1+(\tau\omega)^2]\,. \qquad (13)$$

Things to note:

(i) at frequencies $\omega << 1/\tau$, $M_{dB}(\omega) \cong 20\log_{10}(g_s)$.

(ii) at frequency $\omega = 1/\tau$, $M_{dB}(\omega) = 20\log_{10}(g_s) - 10\log_{10}(2) = 20\log_{10}(g_s) - 3dB$

(iii) at a frequency $\omega_1 >> 1/\tau$, $M_{dB}(\omega_1) \cong 20\log_{10}(g_s) - 20\log_{10}(\tau\omega_1)$

at a frequency $\omega_2 = 10\omega_1$,

$$\begin{aligned} M_{dB}(\omega_2) &\cong 20\log_{10}(g_s) - 20\log_{10}(\tau\omega_2) \\ &= 20\log_{10}(g_s) - 20\log_{10}(\tau 10\omega_1) \\ &= 20\log_{10}(g_s) - 20\log_{10}(\tau\omega_1) - 20\log_{10}(10) \\ &= M_{dB}(\omega_1) - 20dB \end{aligned}$$

In words, the FRF is flat at frequencies much lower than the frequency $1/\tau$, and drops at a rate of 20dB per decade increase at frequencies much higher than the frequency $1/\tau$. Finally, at the frequency $1/\tau$, the magnitude has dropped 3dB relative to its low frequency magnitude. The frequency $1/\tau$ is called the system ***break frequency***, since it is at this frequency that the zero-slope low frequency behavior begins to transition to the high frequency slope of 20dB/decade.

Now, to make this first order system function as a low-pass filter it is only necessary to set $g_s = 1$. Acting as a filter, the break frequency is called the filter ***cut-off*** frequency. Suppose that we desire a cut-off frequency $f_{co} = 10\ Hz.$ Then $\omega_{co} = 20\pi\ rad/\sec$, and the filter time constant is $\tau = 1/\omega_{co} = 1/20\pi = 0.016\sec.$ For this filter: $H(s) = \dfrac{1}{.016s+1}$ we can plot the FRF using two Matlab commands. First, we define the system: 'sys = tf(1,[.016 1])'. Then we type 'bode(sys)'.

The result is given below.

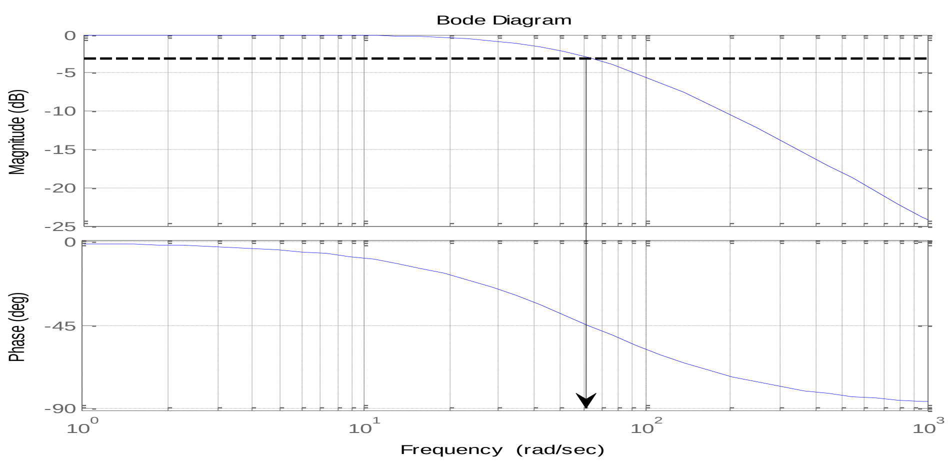

Figure 4. First order low-pass filter FRF with cut-off frequency $f_{co} = 10\ Hz.$ (i.e. $\omega_{co} \cong 62.8 r/s$)

NEW MATERIAL 11/13/19

Application to the Von Karman Wind Model

We are now in a position to address the Von Karman wind model. From Matlab we have:

$\Phi_w(\omega)$	$\dfrac{\sigma_w^2 L_w}{\pi V}\cdot\dfrac{1+\frac{8}{3}\left(1.339L_w\frac{\omega}{V}\right)^2}{\left[1+\left(1.339L_w\frac{\omega}{V}\right)^2\right]^{11/6}}$

For simplicity, we will express the vertical wind velocity *psd* as:

$$\Phi_{w_g}(\omega)=\frac{\sigma_w^2 L_w}{\pi V}\frac{1+\frac{8}{3}\left(\frac{1.339L_w}{V}\right)^2\omega^2}{\left[1+\left(\frac{1.339L_{w_g}}{V}\omega\right)^2\right]^{11/6}}=\frac{\sigma_w^2\tau}{1.339\pi}\quad\frac{1+\frac{8}{3}(\tau\omega)^2}{\left[1+(\tau\omega)^2\right]^{11/16}}.$$

We will now compute $\int_0^\infty \Phi_{w_g}(\omega)d\omega$. To this end, let $x=\tau\omega$. Then $d\omega=(1/\tau)dx$, so that

$$\int_0^\infty \Phi_{w_g}(\omega)d\omega=\frac{\sigma_w^2}{1.339\pi}\int_0^\infty\frac{1+\frac{8}{3}x^2}{(1+x^2)^{11/6}}dx\,.$$

From the Matlab commands: f=@(x)(1+(8/3)*x.^2)./(1+x.^2).^(11/6);   Int=integral(f,0,inf) = 4.2065, we have

$\int_0^\infty \Phi_{w_g}(\omega)d\omega=\dfrac{\sigma_w^2}{1.339\pi}(4.2065)=\sigma_w^2$. Hence, we can conclude that $\Phi_{w_g}(\omega)$ is a **1-sided** *psd*, and that the total power is σ_w^2.

Recall that the Fourier transform theorem states that $\dfrac{1}{2\pi}\int_0^\infty \Phi_{w_g}(\omega)d\omega=\sigma_w^2$. Hence, we have a problem; namely the

$1/2\pi$ that is absent in the integral we computed above. Let's write

$\int_0^\infty \Phi_{w_g}(\omega)\dfrac{d\omega}{2\pi}=\sigma_w^2$. The units of $\dfrac{d\omega}{2\pi}$ are cycles per second (i.e. Hz). Hence, the units of $\Phi_{w_g}(\omega)$ **must** be power per Hz; even though the units of ω are radians per second.

If you're confused, know that you're in good company. In fact, in Grover Brown's book on the subject, he devotes a full paragraph to that "pesky 2π factor'. And in the DOD Interface Standard (MIL-STD-1797A) discussed in class, it is noted on p.820: "Finally, when using the more complex models it seems nearly impossible to formulate a program without an error involving a factor of 2 or pi. The lesson here is to measure the statistics of the output of the disturbance model before starting piloted evaluations."

So, in fact, the above computed integral was not $\int_0^\infty \Phi_{w_g}(\omega)d\omega=\sigma_w^2$. Rather, it was $\int_0^\infty \Phi_{w_g}(\omega)\dfrac{d\omega}{2\pi}=\sigma_w^2$. In words, it was integration in Hz; not in radians per second. It would have been clearer to write $\int_0^\infty \Phi_{w_g}(2\pi f)df=\sigma_w^2$. However, most often we are given $\Phi_{w_g}(\omega)$, not $\Phi_{w_g}(2\pi f)$. And there is good reason for being given $\Phi_{w_g}(\omega)$. For, if one has $\Phi_{w_g}(\omega) = 2|G(i\omega)|^2$ [The factor of 2 is due to the fact that this is a **1-sided** pdf.], then one can readily recover $G(i\omega)$ that is the FRF associated with the transfer function $G(s)$.

So, we can make the following conclusions about $\Phi_{w_g}(\omega) = \frac{\sigma_w^2 L_w}{\pi V} \frac{1+\frac{8}{3}\left(\frac{1.339 L_w}{V}\right)^2 \omega^2}{\left[1+\left(\frac{1.339 L_{w_g}}{V}\omega\right)^2\right]^{11/6}}$:

(C1) It is the 1-sided pdf. Hence, the 2-sided psd is $\Phi'_{w_g}(\omega) = (1/2)\Phi_{w_g}(\omega)$.

(C2) The factor $(1/2\pi)$ has been incorporated in $\Phi_{w_g}(\omega)$.

These conclusions make it difficult to recover $G(s)$ directly from $\Phi_{w_g}(\omega)$. For, in fact:

$\Phi_{w_g}(\omega) = (2)(1/2\pi)|G(i\omega)|^2$. Equivalently, $|G(i\omega)|^2 = \pi\,\Phi_{w_g}(\omega)$. In particular:

$$\pi\,\Phi_{w_g}(\omega) = \frac{\sigma_w^2 L_w}{V} \frac{1+\frac{8}{3}\left(\frac{1.339 L_w}{V}\right)^2 \omega^2}{\left[1+\left(\frac{1.339 L_w}{V}\omega\right)^2\right]^{11/6}} = |G(i\omega)|^2.$$

In order to readily recover $G(s)$, we will make the approximation.

$$\mathbf{c}\frac{\sigma_w^2 L_w}{V} \frac{1+\frac{8}{3}\left(\frac{1.339 L_w}{V}\right)^2 \omega^2}{\left[1+\left(\frac{1.339 L_w}{V}\omega\right)^2\right]^{\mathbf{2}}} = |G(i\omega)|^2$$

The first order of business is to find the value of **c** so that $\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|^2 \, d\omega = \sigma_w^2$.

As before, write:

$$\mathbf{c}\frac{\sigma_w^2 L_w}{V} \frac{1+\frac{8}{3}\left(\frac{1.339 L_w}{V}\right)^2 \omega^2}{\left[1+\left(\frac{1.339 L_{w_g}}{V}\omega\right)^2\right]^{\mathbf{2}}} = \mathbf{c}\frac{\sigma_w^2 \tau}{1.339} \quad \frac{1+\frac{8}{3}(\tau\omega)^2}{\left[1+(\tau\omega)^2\right]^2}.$$

Again, , let $x = \tau\omega$. Then $d\omega = (1/\tau)dx$, so that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{c}\frac{\sigma_w^2 \tau}{1.339} \frac{1+\frac{8}{3}(\tau\omega)^2}{\left[1+(\tau\omega)^2\right]^2} d\omega = \mathbf{c}\frac{2\sigma_w^2}{1.339(2\pi)} \int_0^{\infty} \frac{1+\frac{8}{3}x^2}{\left[1+x^2\right]^2} dx = \sigma_w^2$$

Then, from: f=@(x)(1+(8/3)*x.^2)./(1+x.^2).^2; Int=integral(f,0,inf) = 2.8798

We have: $\mathbf{c}\dfrac{2\sigma_w^2}{1.339(2\pi)}(2.8798)=\sigma_w^2$, so that $\mathbf{c}=\mathbf{1.4607}$. Hence,

$$|G(i\omega)|^2=1.4607\frac{\sigma_w^2 L_w}{V}\frac{1+\frac{8}{3}\left(\frac{1.339L_w}{V}\right)^2\omega^2}{\left[1+\left(\frac{1.339L_{w_g}}{V}\omega\right)^2\right]^2}=\left(\frac{1.4607}{1.339}\sigma_w^2\tau\right)\frac{1+\frac{8}{3}(\tau\omega)^2}{\left[1+(\tau\omega)^2\right]^2}.$$

It should be clear that the numerator is $1+\frac{8}{3}(\tau\omega)^2=(1+i\sqrt{8/3}\,\tau\omega)((1-i\sqrt{8/3}\,\tau\omega)$.

What may not be so clear is that the denominator is: $\left[1+(\tau\omega)^2\right]^2=[(1+i\tau\omega)(1-i\tau\omega)]^2=(1+i\tau\omega)^2(1-i\tau\omega)^2$.

Hence, we find that: $G(s)=\sqrt{\dfrac{1.4607}{1.339}\sigma_w^2\tau}\;\dfrac{1+(\sqrt{8/3}\,\tau)s}{(1+\tau s)^2}$, where $\tau=\dfrac{1.339L_w}{V}$.

For comparison to Matlab's shaping filter, write this as: $G(s)=\sigma_w\sqrt{1.4607\dfrac{L_w}{V}}\left(\dfrac{1+(\sqrt{8/3}\,\tau)s}{(1+\tau s)^2}\right)$.

The Matlab expression is:

$$\frac{\sigma_w\sqrt{\frac{1}{\pi}\cdot\frac{L_w}{V}}\left(1+2.7478\frac{L_w}{V}s+0.3398\left(\frac{L_w}{V}\right)^2 s^2\right)}{1+2.9958\frac{L_w}{V}s+1.9754\left(\frac{L_w}{V}\right)^2 s^2+0.1539\left(\frac{L_w}{V}\right)^3 s^3}$$

The difference in the polynomials is because Matlab did a series expansion. To account for the power 11/6. We simply assumed the power was 2. The most important difference is in the static gains:

Ours is $G(0)=\sigma_w\sqrt{1.4607\dfrac{L_w}{V}}$, while Matlab's is $G(0)_{Matlab}=\sigma_w\sqrt{\dfrac{1}{\pi}\dfrac{L_w}{V}}=\dfrac{1}{\sqrt{1.4607\pi}}\sigma_w\sqrt{1.4607\dfrac{L_w}{V}}=0.4668\sigma_w G(0)$.

In words, while Matlab specifies a value for σ_w, the shaping filter actually uses the value $0.4668\sigma_w$. This can have serious repercussions in simulator testing. If, for example, it is specified that the turbulence standard deviation should be 20 fps, in fact, the actual turbulence will have a standard deviation of less than 10 fps!

END OF NEW MATERIAL 11/13/19

Notice that we can write $1+(\tau\omega)^2=[1+\tau(i\omega)][1-\tau(i\omega)]=[1+\tau(i\omega)]\overline{[1+\tau(i\omega)]}$, where the bar denotes the complex conjugate. If we let $s=i\omega$, we then have

$$\Phi_{w_g}(\omega) = \frac{\sigma_u^2}{\left[1+(\tau\omega)^2\right]^2} = \left[\frac{1}{(\tau s+1)^2}\right]\left[\overline{\frac{1}{(\tau s+1)^2}}\right]\sigma_u^2 \quad \stackrel{\Delta}{=} H(s)\overline{H(s)} \quad \left|H(s)\right|^2 \sigma_u^2$$

where we have defined the "transfer function

$$H(s) \stackrel{\Delta}{=} \frac{1}{(\tau s+1)^2}$$

For any time, *t*, the white noise variance is $\sigma_u^2 = E[u(t)^2]$, where $E(*)$ denotes the *expected value*. Recall from Definition 3 above that $E[|W(i\omega)|^2] \stackrel{\Delta}{=} \Phi_w(\omega)$. Now, write

$$W(s) = H(s)U(s).$$

In words, we are representing the wind, $w(t)$, as the output of the system $H(s)$, with "input" $u(t)$.

We then have

$$\Phi_w(\omega) = E[|W(i\omega)|^2] = E[|H(i\omega)U(i\omega)|^2] = |H(i\omega)|^2 \; E[|U(i\omega)|^2] \quad = \quad |H(i\omega)|^2 \; \sigma_u^2$$

<u>Conclusion</u>: We can simulate Von Karman wind data by running white noise with variance σ_u^2 through the transfer function *H*(*s*).

Table of Laplace and Z Transforms

(Please email me if you find an error) Using this table for Z Transforms with Discrete Indices
Shortened 2-page pdf of Laplace Transforms and Properties

Entry #	Laplace Domain	Time Domain	Z Domain (t=kT)
1	1	$\delta(t)$ unit impulse	1
2	$\frac{1}{s}$	$u(t)$ unit step	$\frac{z}{z-1}$
3	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$
4	$\frac{1}{s+a}$	e^{-at}	$\frac{z}{z-e^{-aT}}$
5		$b^k \quad \left(b=e^{-aT}\right)$	$\frac{z}{z-b}$
6	$\frac{1}{(s+a)^2}$	te^{-at}	$\frac{Tze^{-aT}}{\left(z-e^{-aT}\right)^2}$
7	$\frac{1}{s(s+a)}$	$\frac{1}{a}\left(1-e^{-at}\right)$	$\frac{z\left(1-e^{-aT}\right)}{a(z-1)\left(z-e^{-aT}\right)}$
8	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$\frac{z\left(e^{-aT}-e^{-bT}\right)}{\left(z-e^{-aT}\right)\left(z-e^{-bT}\right)}$
9	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}-\frac{e^{-at}}{a(b-a)}-\frac{e^{-bt}}{b(a-b)}$	
10	$\frac{1}{s(s+a)^2}$	$\frac{1}{a^2}\left(1-e^{-at}-ate^{-at}\right)$	
11	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
12	$\frac{b}{s^2+b^2}$	$\sin(bt)$	$\frac{z\sin(bT)}{z^2-2z\cos(bT)+1}$
13	$\frac{s}{s^2+b^2}$	$\cos(bt)$	$\frac{z(z-\cos(bT))}{z^2-2z\cos(bT)+1}$
14	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin(bt)$	$\frac{ze^{-aT}\sin(bT)}{z^2-2ze^{-aT}\cos(bT)+e^{-2aT}}$

15	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos(bt)$	$\frac{z^2 - ze^{-aT}\cos(bT)}{z^2 - 2ze^{-aT}\cos(bT) + e^{-2aT}}$
16	$\frac{Bs+C}{(s+a)^2+\omega_n^2}$	$e^{-at}\left[B\cos(\omega_n t) + \frac{C-aB}{\omega_n}\sin(\omega_n t)\right]$	