A Brief Technical Note on Correction of Standard Atmospheric Turbulence Models

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Abstract

Atmospheric wind turbulence models, such as the von Karmon and Dryden models, have been used by scientists and engineers for many decades. Throughout this time there has been more than occasional confusion in relation to the scale factors associated with the power spectral density (*psd*) expression for a given model. The purpose of this note is to clear up any such confusion. In doing so, we show that the *psd* expressions for the above models are incorrect. Specifically, we show that they include a factor of $1/\pi$ that should not be included. By erroneously including this factor, a desired turbulence standard deviation will be low by a factor $1/\sqrt{\pi} \cong 0.564$. Lack of awareness of this can lead to a 56% underestimation of flight vehicle responses in related simulations.

Nomenclature

$x(t)$	wide continuous-time sense stationary random process, with the units of *t* are seconds.
$R(\tau)$	autocorrelation function associated with $x(t)$.
σ^2	variance (i.e. total power) of the process $x(t)$.
$S(\omega)$	power spectral density associated with $x(t)$, where the units of ω are radians per second.
$G(s)$	shaping filter transfer function.
$G(i\omega)$	shaping filter frequency response function.
V	forward velocity of a flight vehicle.
L_u	spatial correlation length of the frontal wind turbulence profile.

I. Introduction

The use of shaping filters to simulate atmospheric turbulence has been ongoing for many decades. Two models have been particularly popular in relation to flight vehicles. They include the von Karmon and the Dryden models. These models are given in the current *Matlab* aero-block library [1]. They have been included in flight vehicle dynamics textbooks, such as [2] and [3]. As popular as these models are, and have been for many decades, there continues to be elements of confusion in relation to them. In the military documentation [4], it is noted in APPENDIX A (p.694):

> "Finally, when using the more complex models it seems nearly impossible to formulate a program without an error involving a factor of 2 or π. The lesson here is to measure the statistics of the output of the disturbance model before starting piloted evaluations."

In [5] the authors devote an entire section (3.8 on pp.118-120) to what they call "The Pesky 2π Problem", in order to clear up confusion the reader might have.

In this work we hope to alleviate such confusion. It is confusion that is, to a degree, justified, in view of the fact that, as we will now proceed to show, those models are incorrect.

II. The autocorrelation function and the Power Spectral Density

Let $x(t)$ be a regular wide sense stationary (*wss*) random process with autocorrelation function $R(\tau) = E[x(t)x(t+\tau)]$. The associated *psd* is given by:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau \,. \tag{2.1}$$

From the theory of Fourier transform pairs, (specifically, the Wiener-Kinchine Theorem [5]), we then have

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \,. \tag{2.2}$$

We will assume that $E[x(t)] = 0$. Then $R(0) = \sigma^2$, so that (2.2) gives

$$\sigma^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega \,. \tag{2.3}$$

Typically, σ^2 is referred to as the power of the process $x(t)$. What (2.3) shows is that this power can be computed by integrating the *psd*.

It is at this point that one must take care to appreciate the units of $S(\omega)$. To this end, write (2.3) as

$$\sigma^2 = \int_{-\infty}^{\infty} S(\omega)\frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} S(2\pi f)df \tag{2.4}$$

where $f = \omega/2\pi$ is the circular frequency associated with the radial frequency ω. In this work we will assume that the units of ω are radians/second. It follows that the units of $f = \omega/2\pi$ are cycles/second; units that are commonly referred to as Hertz (*Hz*). Hence, from (2.4), it is clear that $S(\omega) = S(2\pi f)$ must be in units of *Hz*.

As noted above, this can be a source of confusion. After all, the psd is usually written as $S(\omega)$; not as $S(2\pi f)$. So it is reasonable to assume that $S(\omega)$ is the power per rad/sec. It is of central importance in this work. Hence, we will refer to it as:

FACT 1: The units of the *psd* $S(\omega)$ are power/Hz.

A second potential source of confusion is related to the fact that $S(\omega)$ is the 2-sided *psd*; that is, it is defined over $(-\infty < \omega < \infty)$. However, since $S(\omega) = S(-\omega)$, one can define the 1-sided psd $S_1(\omega) = 2S(\omega)$ for $\omega > 0$. It should be noted that $S_1(0) = S(0)$. However, if we assume that $S(\omega)$ is continuous, as we will here, then $S(0)$ contributes nothing to σ^2. Hence, we need only consider $S_1(\omega) = 2S(\omega)$. The 1-sided *psd* is appealing in that the power is placed at only positive frequencies. Some people have an aversion to dealing with negative frequencies. The distinction between the 1-sided and 2-sided *psd*s is also central to this work, and so will be referred to as:

FACT 2: The 1-sided psd is $S_1(\omega) = 2S(\omega)$.

III. Correction of the Standard Wind Turbulence Models

In this section we will address only one of the many models related to von Karmon and Dryden turbulence provided in Matlab, and given in many books on the subject. It suffices to address only one model, as all the models have the same fault. The Dryden frontal wind turbulence model for an air vehicle traveling at a nominal speed *V* is given by (MIL-F-8785C) [1]:

$$\Phi_u(\omega) = \sigma_u^2\left(\frac{2L_u}{\pi V}\right)\frac{1}{1+\left(L_u\frac{\omega}{V}\right)^2}. \tag{3.1}$$

We will now show that (3.1) is incorrect. Specifically, the factor $1/\pi$ should not be included in (3.1). To this end, we will compute:

$$\int_0^\infty \Phi_u(\omega)d\omega \cdot \tag{3.2}$$

Letting $x = L_u \frac{\omega}{V}$ in (3.1), and noting that $d\omega = \frac{V}{L_u} dx$ in (3.1) allows (3.2) to be expressed as:

$$\int_0^\infty \Phi_u(\omega)d\omega = \sigma_u^2 \left(\frac{2}{\pi}\right) \int_0^\infty \frac{1}{1+x^2} dx = \sigma_u^2 \left(\frac{2}{\pi}\right) \tan^x \Big|_{x=0}^\infty = \sigma_u^2 . \tag{3.3}$$

At this point one might argue that (3.2) is correct, since the total power of the *psd* is, indeed, σ^2. However, recall that (3.2) does not include the factor $1/2\pi$ as was needed in (2.3). Also, as is clear by the integration limits in (3.2), the integrand must have also included a factor of 2 (i.e. it is a 1-sided *psd*). Multiplying by 2, and erroneously dividing $S(\omega)$ [it is already in units of *Hz*] by 2π amounts to a division by π. It follows that the true 2-sided psd is:

$$S_u(\omega) = \sigma_u^2 \left(\frac{2L_u}{V}\right) \frac{1}{1+\left(L_u \frac{\omega}{V}\right)^2} . \tag{3.4}$$

To verify (3.4) write:

$$\frac{1}{2\pi} \int_{-\infty}^\infty S_u(\omega)d\omega = \frac{1}{2\pi} \int_{-\infty}^\infty \sigma_u^2 \left(\frac{2L_u}{V}\right) \frac{1}{1+\left(L_u \frac{\omega}{V}\right)^2} d\omega = \frac{2\sigma_u^2}{\pi} \left(\frac{L_u}{V}\right) \int_0^\infty \frac{1}{1+\left(L_u \frac{\omega}{V}\right)^2} d\omega = \frac{2\sigma_u^2}{\pi} \int_0^\infty \frac{1}{1+x^2} dx = \sigma_u^2 . \tag{3.5}$$

IV. Implications in Relation to the Shaping Filter

Assuming that the random process $x(t)$ is simulated by running continuous-time *unit-intensity* white noise $w(t)$ through a filter whose transfer function is $G(s)$, it follows that

$$S(\omega) = \left|G(i\omega)\right|^2 . \tag{4.1}$$

For a *psd* of the form

$$S(\omega) = \frac{c^2}{\omega_{br}^2 + \omega^2} = \left|G(i\omega)\right|^2 \tag{4.2}$$

where ω_{br} refers to the -3dB break frequency, the filter transfer function is:

$$G(s) = \frac{c}{s + \omega_{br}}. \tag{4.3}$$

From any table of Fourier transform pairs, one finds that the autocorrelation function associated with (4.2) is:

$$R(\tau) = \left(\frac{c^2}{2\omega_{br}}\right) e^{-\omega_{br}|\tau|}. \tag{4.4}$$

In particular:

$$\sigma^2 = R(0) = \left(\frac{c^2}{2\omega_{br}}\right). \tag{4.5}$$

Comparing (4.2) to (3.4) with $c = \sigma_u \sqrt{\frac{2V}{L_u}}$ and $\omega_{br} = \frac{V}{L_u}$, and substituting these into the right side of (4.5) gives the left side. Hence, this is yet another proof of (3.4).

It also gives the specific expression for the shaping filter (4.3):

$$G(s) = \frac{\sigma_u \sqrt{\frac{2V}{L_u}}}{s + \frac{V}{L_u}}. \tag{4.6}$$

Clearly, (4.6) may be expressed as:

$$G(s) = \frac{\sigma_u \sqrt{\frac{2L_u}{V}}}{1 + \frac{L_u}{V} s}. \tag{4.7}$$

The form (4.7) is nearly the same as the shaping filter given in *Matlab* [1]; namely

$$G_M(s) = \frac{\sigma_u \sqrt{\frac{2L_u}{\pi V}}}{1 + \frac{L_u}{V} s}. \tag{4.8}$$

From (4.7) and (4.8) we have:

$$G_M(s) = \frac{\left(\frac{1}{\sqrt{\pi}}\sigma_u\right)\sqrt{\frac{2L_u}{V}}}{1+\frac{L_u}{V}s} = \frac{1}{\sqrt{\pi}}G(s)\cdot \tag{4.9}$$

Hence, we see that for a desired standard deviation σ_u, the Matlab shaping filter actually uses $\sigma_u/\sqrt{\pi}$.

V. A Numerical Simulation

Here, we will assume, for convenience, that $\sigma_u = 1$ and that $\omega_{br} = \frac{V}{L_u} = 1$. Then (4.6) is $G(s) = \frac{\sqrt{2}}{s+1}$. We will assume a sampling period $\Delta = 0.01\text{sec}$. The white noise is obtained by sampling from a normal distribution with mean zero and standard deviation $1/\sqrt{\Delta}$. Using this as the input to the filter resulted in the following plots in Figure 1. The Matlab code use for obtain the simulations is included in the Appendix.

The plot in Fig. 1(a) shows a single simulation. The plot in Fig. 1(b) shows the simulation-based probability density function (*pdf*) of the sample standard deviation $\hat{\sigma}_u$, based on 10^4 simulations.

Both plots provide support for the correctness of the shaping filter (4.7). The support in Fig. 7(a) is anecdotal. The range of the turbulence falls within the ± 3 range. The support provided in Fig. 7(b) is more quantitative. The mean $\mu_{\hat{\sigma}_u} = 0.985$ is within 2% of the theoretical standard deviation. The small difference is due to the fact that the continuous-time process was sampled. What is, perhaps, more interesting is the standard deviation $\sigma_{\hat{\sigma}_u} = 0.070$. From this, one could expect that any single simulation would have a sample standard deviation that will likely fall within the range $[0.8, 1.2]$. Even so, one should not be overly surprised, were it to occasionally fall outside of this range.

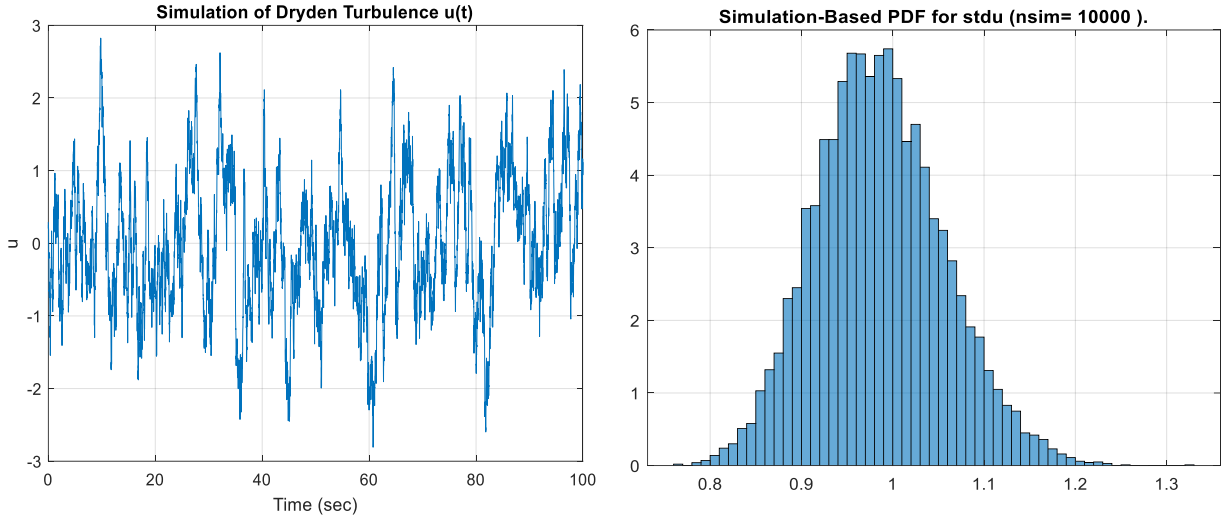

Fig. 1 (a) A single simulation with $\hat{\sigma}_u = 0.923$. **(b)** The pdf for $\hat{\sigma}_u$ based on 10^4 simulations. Its mean is $\mu_{\hat{\sigma}_u} = 0.985$, and its standard deviation is $\sigma_{\hat{\sigma}_u} = 0.070$.

VI. Summary and Conclusions

This work addressed the correctness of a class of turbulence simulation models that have been in use for many decades. This class includes the von Karmon and Dryden models for simulating various turbulence profiles. It was noted that those models have been used in many textbooks, as well as in military documentation and in the most recent version of *Matlab*. It was then shown that those *psd* models are incorrect. They are off by a factor of $1/\pi$. This error was attributed to (i) incorporating a 1-sided *psd*, and (ii) incorrectly including the factor $1/2\pi$ into the *psd* whose units were already *Hz*. Consequently, the associated shaping filters are off by a factor of $1/\sqrt{\pi}$.

The Dryden frontal wind turbulence model was then used to demonstrate their incorrectness. Finally, a numerical example was provided in order to not only add further validation to the correct *psd* (3.4) , but also to quantify the potential range of variability of the sample standard deviation for any given simulation. Without such quantification, one might be tempted to doubt the correctness of (3.4) (or any of the other models on *Matlab* [1]), after running a single simulation.

In conclusion, it is our hope that this work will help to clear up any future confusion in relation to factors such as 2 and π, encountered when using these models. Furthermore, it is hoped that future textbooks, military documentation, journal articles, and versions of *Matlab* will make appropriate corrections. For, it could be extremely troublesome to realize that a simulator validation of a given flight vehicle utilized a turbulence power that was a factor of π smaller than was specified.

Acknowledgement

I am indebted to the students I have taught in my flight dynamics and feedback controls classes at Iowa State University. For, without their thoughtful requests for a more rigorous explanation of this topic, I might never have produced this work.

References

[1] Matlab version R2018a.
https://www.mathworks.com/help/aeroblks/drydenwindturbulencemodelcontinuous.html

[2] Nelson, R.C. *Flight Stability and Automatic Control*, 2nd ed., McGraw Hill (1998).

[3] Schmidt, D.K. *Modern Flight Dynamics*, McGraw Hill (2012).

[4] Department of Defense Handbook: Flying Qualities of Piloted Aircraft. MIL-HBDK 1797 (19 Dec.1997).

[5] Brown, R.G. and Hwang, P.Y.C. *Introduction to Random Signals and Applied Kalman Filtering*, 4th ed., Wiley (2012).

Appendix Matlab code used in the Example in Section 5

```
%TechNoteExample.m
G=tf(sqrt(2),[1 1]);
fbw=1/(2*pi);
del=0.01;
ntot=15000;
n=10000;
nsim=10^4;
STDu=zeros(1,nsim);
for k=1:nsim
   w=normrnd(0,1/sqrt(del),1,ntot);
   t=0:del:(ntot-1)*del;
   u=lsim(G,w,t);
   u=u(ntot-n+1:ntot);
   STDu(k)=std(u);
end
t=0:del:(n-1)*del;
figure(1)
histogram(STDu,'Normalization','pdf')
title(['Simulation-Based PDF for stdu (nsim= ',num2str(nsim),' ).'])
grid
muSTDu=mean(STDu);
stdSTDu=std(STDu);
[muSTDu stdSTDu]
figure(2)
plot(t,u)
title('Simulation of Dryden Turbulence u(t)')
xlabel('Time (sec)')
ylabel('u')
grid
STDu(nsim)
```