Homework 6 AERE355 Fall 2019 Due 11/20 (W) SOLUTION

PROBLEM 1(30pts) The transfer function of a certain first order *Low Pass Filter* (*LPF*) is $H(s)=\dfrac{1}{(s/\omega_{BW})+1}$ where the frequency range $[0,\omega_{BW}]$ is called the filter $-3dB$ bandwidth (*BW*).

(a)(8pts) The dc (i.e. very low frequency) gain $H(i0)=1$ gives $20\log\left[|H(i0)|\right]=20\log(1)=0\,dB$. Use the same computation to determine why the interval $[0,\omega_{BW}]$ is called the $-3dB$ *bandwidth*.

<u>Solution</u>: $H(i\omega_{BW})=\dfrac{1}{1+1i}$ gives $|H(i\omega_{BW})|=\dfrac{1}{\sqrt{1^2+1^2}}=2^{-1/2}$. Hence: $20\log\left[|H(i\omega_{BW})|\right]=20\log\left(2^{-1/2}\right)=-10\log(2)=-3.01dB$.

The interval $[0,\omega_{BW}]$ is the range of frequencies such that the magnitude of $H(i\omega)$ is with 3dB of its static gain.

(b)(6pts) The 'bode(H)' command was used to obtain the *Frequency Response Function* (*FRF*) shown at right for $\omega_{BW}=100$ rad/s. For an input $u(t)=\sin(1000t)$ use the information in Figure 1(b) to estimate the values of M and θ expression for the ***steady state*** output $y(t)=M\sin(1000t+\theta)$.

<u>Solution</u>:

$M(1000)_{dB}\cong -20dB \Rightarrow M(1000)=10^{-1}=0.1$.

$\theta(1000)\cong -83^o=-1.45rad$

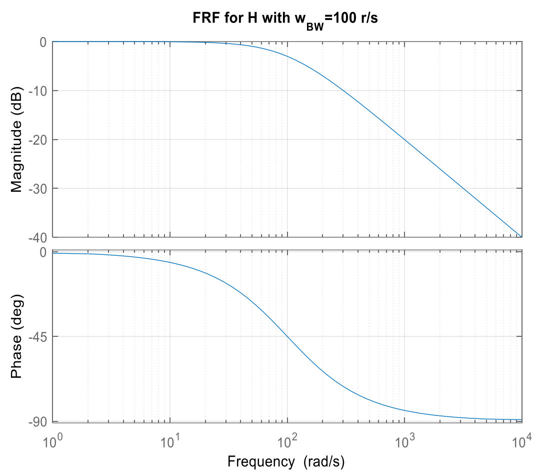
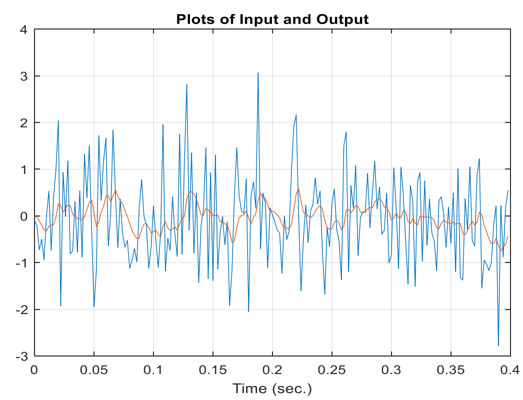


Figure 1(b) FRF for $H(s)$ with

(c)(10pts) The term *low pass filter* is due to the fact that when an input is supplied to $H(s)$, the output is a filtered version of the input. Specifically, lower frequency (meandering) components are allowed to pass freely, while higher frequency (rapidly jittering) components are suppressed. To visualize this, use the 'lsim' command to complete the code at 1(c) in relation to a measured *white noise* input $u(t)$. This is an array of numbers randomly sampled from a normal distribution having mean 0 and standard deviation 1.0. (i) Overlay plots of the input and output, and (ii) discuss how they differ.

<u>Solution</u>: [See code @ 1(c).]



It is clear that the output is far less jittery than the input.

Figure 1(c) Plots of input and output.

(d)(6pts) At frequencies $\omega >> \omega_{BW}$ you should see that your FRF in (b) has the following properties: (P1) the magnitude has a slope of 20 dB/decade; (P2) the phase is ~-90°. Using $H(s)=\dfrac{1}{(s/\omega_{BW})+1}$, prove this for arbitrary ω_{BW}.

<u>Solution</u>: $H(i\omega)\big|_{\omega/\omega_{BW}>>1}=\dfrac{1}{i(\omega/\omega_{BW})+1}\Bigg|_{\omega/\omega_{BW}>>1}\cong\dfrac{1}{i(\omega/\omega_{BW})}=\dfrac{-i}{(\omega/\omega_{BW})}$. Hence, $\theta(\omega)\cong -90^o$.

Also, $M(\omega)=\dfrac{1}{(\omega/\omega_{BW})}$ and $M(10\omega)=\dfrac{1}{(10\omega/\omega_{BW})}=\dfrac{1}{10}M(\omega)$, so that $M(10\omega)dB=20\log\left(\dfrac{1}{10}M(\omega)\right)=20\log\left(M(\omega)\right)dB-20dB$.

Hence, the magnitude is dropping at a rate of 20dB/decade.

PROBLEM 2(35pts) The dynamics of a plane's short period response to wind can be approximated by:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha / u_0 & 1 \\ M_\alpha + M_{\dot{\alpha}} Z_\alpha / u_0 & M_{\dot{\alpha}} + M_q \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} -Z_\alpha / u_0^2 & 0 \\ -M_\alpha / u_0 & -M_q \end{bmatrix} \begin{bmatrix} w_g \\ q_g \end{bmatrix}. \tag{2.1}$$

Consider the Boeing 747 @ M=.9.

(a)(10pts) Obtain the two transfer functions $H_{w_g}^{(\alpha)}(s)$ and $H_{w_g}^{(q)}(s)$ by forming the (A,B,C,D) matrices and then using the ss2tf command.

Solution: [See code @ 2(a).] $H_{w_g}^{(\alpha)}(s) = \dfrac{.000452\,s + .00210}{s^2 + .9331s + 1.775}$ and $H_{w_g}^{(q)}(s) \cong \dfrac{.00186s}{s^2 + .9331s + 1.775}$.

(b)(5pts) If you convert $H_{w_g}^{(\alpha)}(s)$ to have units [degrees/fps] you should have $H_{w_g}^{(\alpha)}(s) \cong \dfrac{0.026\,s + 0.12}{s^2 + 0.933s + 1.775}$. Use the Matlab 'bode' command to obtain a plot of the corresponding FRF.

Solution: [See code @ 2(b).]

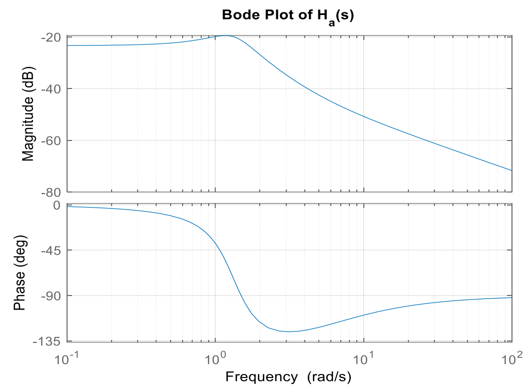


Figure 2(b) Plot of $H_{w_g}^{(\alpha)}(i\omega)$.

(c)(5pts) In (6.35) on p.220 the authors describe a sharp edge gust as $w_g(t) = A_g u_s(t)$. Suppose that the plane encounters such a gust with amplitude of 50 mph (i.e. 73.33 fps). Use the Matlab command 'step' to obtain a plot of the response $\alpha(t)$ (in degrees) to this input.

Solution: [See code @ 2(c).]

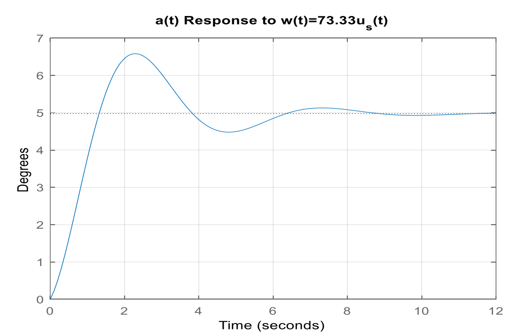


Figure 2(c) Plot of $\alpha(t)$.

(d)(5pts) Since the Laplace transform of $w_g(t) = A_g u_s(t)$ is $w_g(s) = A_g / s$, the Fourier transform is $w_g(i\omega) = -i(A_g / \omega)$. It follows that

$\Phi_{w_g}(\omega) \overset{\Delta}{=} |w_g(i\omega)|^2 = (A_g / \omega)^2$ describes the distribution of the gust power as a function of frequency. In relation to the gust in (c), obtain a plot of $\Phi_{w_g}(\omega)_{dB} = 10\log_{10}\Phi_{w_g}(\omega)$ over the range of frequencies in Figure 2(b). Use the 'semilogx' command in relation to your frequency axis so that it has the same range as that in Figure 2(b).

Solution: [See code @ 2(d).]

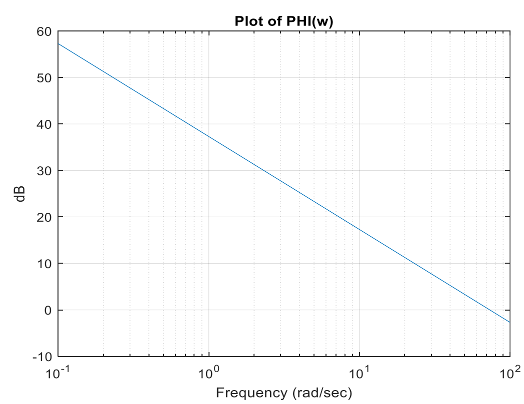


Figure 2(d) Plot of $\Phi_{w_g}(\omega)_{dB}$.

(e)(5pts) From Figure 2(b) $H_{w_g}^{(\alpha)}(i\omega) \cong \frac{0.026(i\omega)+0.12}{-\omega^2+0.933(i\omega)+1.775}$. Hence, the spectral power associated with the response $\alpha(t)$ is:
$\Phi_\alpha(\omega) = |H_{w_g}^{(\alpha)}|^2 \Phi_{w_g}(\omega)$. Plot this in dB over the range of frequencies in Figure 2(b).
Solution: [See code @ 2(e).]

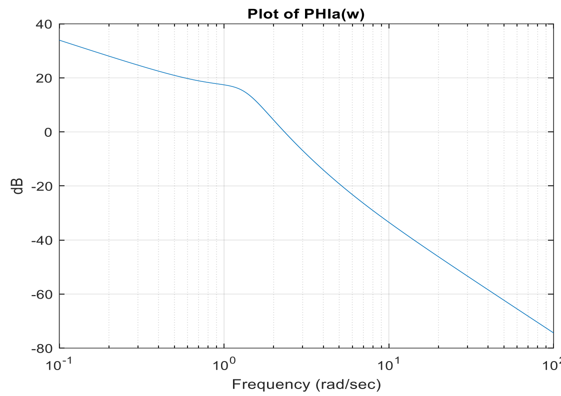


Figure 2(e) Plot of $\Phi_\alpha(\omega)_{dB}$.

(f)(5pts) The total power associated with $\alpha(t)$ is $PWR = \int_0^\infty \alpha^2(t)dt$. It can also be computed as $PWR = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\alpha(\omega)d\omega$. This equivalence is known as *Parseval's Theorem*. Show that the total power is infinite.
Solution:

From Figure 2(b) we see that $\alpha(t) \cong 5^o$ for $t > 10\,\text{sec}$. Hence, clearly $PWR = \int_0^\infty \alpha^2(t)dt = \infty$.

PROBLEM 3(35pts) In this problem we consider von Karmon turbulence $w_g(t)$. From p.228 of Nelson, the vertical wind turbulence spatial power spectral density (*psd*) is ***almost*** [i.e. (6.54) is incorrect] given by:

$$\Phi_{w_g}(\Omega) = \sigma_w^2 L_w \frac{1+\frac{8}{3}(1.339L_w\Omega)^2}{\left[1+(1.339L_w\Omega)^2\right]^{11/6}}. \qquad (6.54') \qquad ; \qquad \Phi_{w_g}(\omega) = \sigma_w^2\left(\frac{L_w}{u_0}\right)\frac{1+\frac{8}{3}(1.339L_w/u_0)^2\omega^2}{\left[1+(1.339L_w/u_0)^2\omega^2\right]^{11/6}}. \qquad (3.1)$$

Equation (3.1), as well as the u_0 factor, follows from the fact that $\Omega = \omega/u_0$ [see (6.55)] and that $d\Omega = d\omega/u_0$.

The total power of the random process $w_g(t)$ is its *variance*: $\sigma_w^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{w_g}(\omega)d\omega$.

(a)(5pts) Use the change of variable theorem and the command 'integral' to prove that this is the case. Specifically, show that $\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{L_w}{u_0}\right)\frac{1+\frac{8}{3}(1.339L_w/u_0)^2\omega^2}{\left[1+(1.339L_w/u_0)^2\omega^2\right]^{11/6}}d\omega = 1$.

Solution: [See code @ 3(a).] Let $x = (1.339L_w/u_0)\omega$. Then $d\omega = dx/(1.339L_w/u_0)$, so that using the change of variable theorem, the above integral is: $\frac{1}{2\pi(1.339)}\int_{-\infty}^{\infty}\frac{1+(8/3)x^2}{(1+x^2)^{11/6}}dx = \frac{1}{\pi(1.339)}\int_{0}^{\infty}\frac{1+(8/3)x^2}{(1+x^2)^{11/6}}dx = \frac{1}{2\pi(1.339)}(8.4131) = 1$.

(b)(5pts) For convenience, we will approximate (3.1) as $\Phi_{w_g}(\omega) = \sigma_{w_g}^2\left(\frac{L_w}{u_0}\right)(1.4607)\frac{1+\frac{8}{3}(1.339L_w/u_0)^2\omega^2}{\left[1+(1.339L_w/u_0)^2\omega^2\right]^2}$.

Show that the number 1.4607 is needed so that $\sigma_w^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{w_g}(\omega)d\omega$.

Solution: [See code @ 3(b).] $\int_{-\infty}^{\infty}\frac{1+(8/3)x^2}{(1+x^2)^2}dx = 5.7596$, so that $\frac{1}{2\pi(1.339)}(5.7596) = 0.6846$. In order for this to equal 1.0 we need to multiply it by $1/0.6846 = 1.4607$.

(c)(5pts) In order to simulate turbulence, we need to obtain the ***shaping filter***, $G(s)$, such that $\Phi_{w_g}(\omega) = \sigma_{w_g}^2\left|G(i\omega)\right|^2$.

Notice that from (3.1) $\left|G(i\omega)\right|^2$ has the form $|G(i\omega)|^2 = a^2\frac{1+(b\omega)^2}{\left[1+(c\omega)^2\right]^2}$. For this form, show that $G(s) = a\frac{1+bs}{(1+cs)^2}$.

Solution: $|G(i\omega)|^2 = a^2\frac{|1+ib\omega|^2}{|(1+ic\omega)^2|^2} = a^2\frac{1+(b\omega)^2}{(1+ic\omega)^2(1-ic\omega)^2} = a^2\frac{1+(b\omega)^2}{[(1+ic\omega)(1-ic\omega)]^2} = a^2\frac{1+(b\omega)^2}{[1+(c\omega)^2]^2}$.

(d)(10pts) For $L_w = 325\,ft$ and $u_0 = 871\,fps$ it can be shown from (c) that $G(s) \cong \frac{0.602s+0.738}{0.245s^2+0.999s+1}$. For wind gust variance $\sigma_{w_g}^2 = 20^2$ arrive at a plot of $\Phi_{w_g}(\omega) = \sigma_{w_g}^2\left|G(i\omega)\right|^2$.

Solution: [See code @ 3(d).]

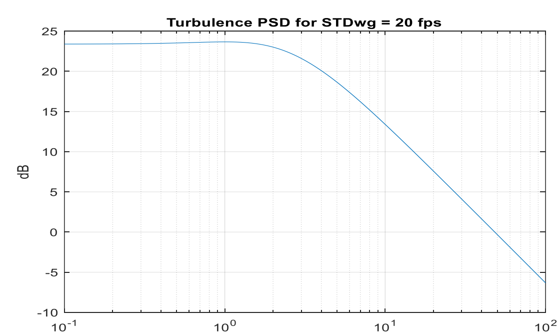


Figure 3(d) Plot of $\Phi_{w_g}(\omega) = \sigma_{w_g}^2\left|G(i\omega)\right|^2$.

(e)(10pts) For the shaping filter and variance given in (d) use the 'lsim' command to simulate turbulence $w_g(t)$ for $t = 0:\Delta:10$, where the sampling period is $\Delta = \pi/100$. Then, to validate your simulation note whether it remains in the $\pm 3\sigma_{w_g}$. [**Note**: The input white noise $u(t)$ to the filter [[obtained using the normrnd command] must have standard deviation $\sigma_u = 1/\sqrt{\Delta}$.]

<u>*Solution*</u>: [See code @ 3(e).]

From the plot it is clear that the simulation remains in the $\pm 3\sigma_{w_g} = \pm 60$ range.

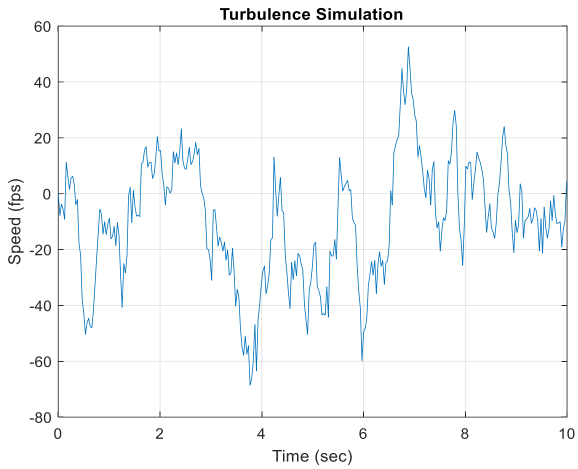


Figure 3(b) von Karmon turbulence simulation.

Appendix Matlab Code

```
%PROGRAM NAME: hw6.m
%PROBLEM 1:
%(b):
wBW=100;
s=tf('s');
H=1/((s/wBW) + 1);
figure(10)
bode(H)
title('FRF for H with w_B_W=100 r/s')
grid
%--------------
%(c):
rng('shuffle') %This will ensure that no two students have the same numbers.
n=200;
u=normrnd(0,1,n,1);
dt=0.002;
t=0:n-1; t=dt*t';
y=lsim(H,u,t);
figure(11)
plot(t,[u,y])
title('Plots of Input and Output')
xlabel('Time (sec.)')
grid
%=====================================================
%PROBLEM 2
%(a):
% Plane Information:
S=5500; b=195.68; cbar=27.31; W=636600; Iy=33.1e6;
CLa=5.5; CD=0.042; Cma=-1.6; Cmadot=-9.0; Cmq=-25.0;
%Altitude Information:
rho=5.8727e-4; a=968.08;
u0=0.9*a; Q=0.5*rho*u0^2; m=W/32.17;
%--------------
Za=-(CLa+CD)*Q*S/m;
Ma=Cma*Q*S*cbar/Iy;
Madot=Cmadot*Q*S*cbar^2/(2*u0*Iy);
Mq=Cmq*Q*S*cbar^2/(2*u0*Iy);
% Compute (A,Bw,C,D) Matrices:
A=[Za/u0 , 1 ; Ma+Madot*Za/u0 ,Madot+Mq];
B=[-Za/u0^2,0;-Ma/u0,-Mq];
C=eye(2); D=zeros(2,2);
%Transfer functions related to wg:
[Hn,Hd]=ss2tf(A,B,C,D,1); %Transfer functions re: wg;
Ha=tf(Hn(1,:),Hd)
Hq=tf(Hn(2,:),Hd)
%(b):
Had=(180/pi)*Ha
figure(20)
bode(Had)
title('Bode Plot of H_a(s)')
grid
%(c):
Ag=73.33;
figure(21)
step(Ag*Had)
title('a(t) Response to w(t)=73.33u_s(t)')
ylabel('Degrees')
grid
%d):
wvec=logspace(-1,2,1000);
PHIwg=(Ag*wvec.^-1).^2;
PHIwgdB=10*log10(PHIwg);
figure(22)
semilogx(wvec,PHIwgdB)
title('Plot of PHIwg(w)')
xlabel('Frequency (rad/sec)')
ylabel('dB')
```

```
grid
%(e):
Haw=(0.12+1i*.026*wvec)./(1.775-wvec.^2+1i*.933*wvec);
Haw2=abs(Haw).^2;
PHIa=Haw2.*PHIwg;
PHIadB=10*log10(PHIa);
figure(23)
semilogx(wvec,PHIadB)
title('Plot of PHIa(w)')
xlabel('Frequency (rad/sec)')
ylabel('dB')
grid
%============================================================
%PROBLEM 3
%(a):
f=@(x) (1+(8/3)*x.^2)./(1+x.^2).^(11/6);
Qa=2*integral(f,0,inf);
%(b):
f=@(x) (1+(8/3)*x.^2)./(1+x.^2).^2;
Qb=2*integral(f,0,inf);
cb=Qa/Qb; %=1.4607
%(c):
Lw=325; STDwg=20; VARwg=STDwg^2;
a=sqrt(cb*Lw/u0); c=1.339*Lw/u0; b=sqrt(8/3)*c;
G=tf(a*[b 1],[c^2 2*c 1])
s=tf('s');
G=a*(1+b*s)/(1+c*s)^2;
[n,d]=tfdata(G,'v');
f=@(w) abs((n(2)*1i*w+n(3))./(d(1)*(1i*w).^2+d(2)*1i*w+d(3))).^2;
Q=2*integral(f,0,inf)/(2*pi); %Check that Q=1.0
w=logspace(-1,2,1000);
PHIwg=VARwg*abs((n(2)*1i*w+n(3))./(d(1)*(1i*w).^2+d(2)*1i*w+d(3))).^2;
PHIwgdB=10*log10(PHIwg);
figure(30)
semilogx(w,PHIwgdB)
title(['Turbulence PSD for STDwg = ',num2str(STDwg),' fps'])
xlabel('Frequency (r/s)')
ylabel('dB')
grid
%(e):
del=pi/100;
t=0:del:10;
nt=length(t);
u=normrnd(0,1/sqrt(del),1,nt);
wgsim=lsim(STDwg*G,u,t);
figure(31)
plot(t,wgsim)
title('Turbulence Simulation')
xlabel('Time (sec)')
ylabel('Speed (fps)')
grid
```