Homework 3 AERE355 Fall 2019 Due 10/1(F) SOLUTION

PROBLEM 1(25pts) Consider the force equations in Table 3.1 on p.105 of Nelson

(a1) $X - mg\sin\theta = m(\dot{u} + qw - rv)$; (a2) $Y + mg\cos\theta\sin\phi = m(\dot{v} + ru - pw)$; (a3) $Z + mg\cos\theta\cos\phi = m(\dot{w} + pv - qu)$
Suppose that all the <u>linear</u> accelerations $\dot{u} = \dot{v} = \dot{w} = 0$. Then we have:

(b1) $X - mg\sin\theta = m(qw - rv)$; (b2) $Y + mg\cos\theta\sin\phi = m(ru - pw)$; (b3) $Z + mg\cos\theta\cos\phi = m(pv - qu)$.

(a)(6pts) Suppose the pitch rate $q = 0$. Find the two conditions on (u, v, w), one trivial and the other nontrivial, that will make the right sides of (b) all zero.

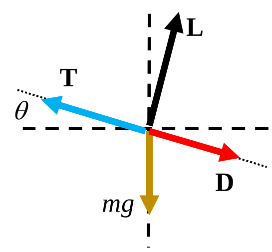
<u>*Solution*</u>: For $q = 0$, then the right sides of (b1) and (b3) will be 0 for $v = 0$. The right side of (b2) will be zero for $ru = pw$, and so $w = (r/p)u$.

(b)(5pts) Suppose that the conditions found in part (a) hold, so that the right side of each of these three equations is zero. We then have: (c1) $X - mg\sin\theta = 0$; (c2) $Y + mg\cos\theta\sin\phi = 0$; (c3) $Z + mg\cos\theta\cos\phi = 0$.
Write $\mathbf{F} = X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k}$. Show that $|\mathbf{F}| = mg$.

<u>*Solution*</u>: $|\mathbf{F}| = mg\sqrt{\sin^2\theta + \cos^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi} = mg\sqrt{\sin^2\theta + \cos^2\theta(\sin^2\phi + \cos^2\phi)} = mg$

(c)(10pts) There are four major forces acting on the airplane: gravity force *mg*, thrust *T*, lift *L*, and drag *D*. For *T* at an angle θ, use a free body diagram to obtain $\mathbf{F} = F_X\,\mathbf{i}_E + F_Z\,\mathbf{k}_E$ in terms of these four forces and θ.

<u>*Solution*</u>: $\mathbf{F} = [(T - D)\cos\theta - L\sin\theta]\mathbf{i}_E + [mg + (D - T)\sin\theta - L\cos\theta]\mathbf{k}_E$



(d)(4pts) Verify your answer in (c) by setting $\theta = 0^o$. Explain, in words, whether or not you feel your expression for **F** makes sense.

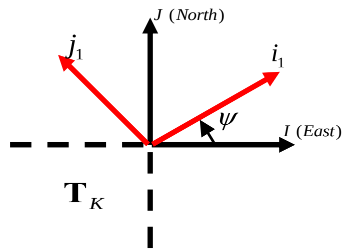
<u>*Solution*</u>: $\mathbf{F} = [T - D]\mathbf{i}_E + [mg - L]\mathbf{k}_E$.

<u>*Explanation*</u>: This makes perfect sense. Lift counters gravity, and drag counters thrust.

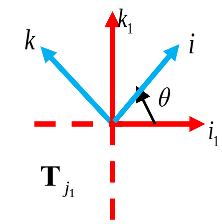
PROBLEM 2(25pts) A mini-drone is to be used in relation to search-and-rescue operations in buildings. Let the drone orientation in the building frame of reference be $[I,J,K]^{tr}=[East, North, UP]^{tr}$ [North is to the left of East], and let its orientation in relation to its body coordinates be $[i,j,k]^{tr}=$[Along the body (b) toward the nose, toward the **left** wing, toward the nose-**up**]^{tr}. [Note: this differs from the traditional body coordinates.]

(a)(15pts) Define the Euler matrix, $\mathbf{T}=\mathbf{T}_{j_1}\mathbf{T}_K$ comprised of two rotations, such that $[i, j, k]^{tr}=\mathbf{T}[I, J, K]^{tr}$, where $\mathbf{T}_K$ denotes the first rotation about the K-axis, and $\mathbf{T}_{j_1}$ denotes the second rotation about the j_1-axis. (i) Use the axes at right to sketch the transformations $\mathbf{T}_K$ and $\mathbf{T}_{j_1}$. [Label axes prior to each rotation, and let ψ and θ denote the *azimuth* and *elevation*, respectively].

Then (ii) from your plots, obtain the expressions for the rotation matrices $\mathbf{T}_K$, $\mathbf{T}_{j_1}$, and finally, (iii) Compute **T**.



1(a.1) $\mathbf{T}_K$ rotation.



1(a.2) $\mathbf{T}_{j_1}$ rotation.

Figure 1(a) Rotation sketches.

Solution: (i) See plots at right.

(ii) $$\begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix}=\mathbf{T}_K\begin{bmatrix} I \\ J \\ K \end{bmatrix}=\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} I \\ J \\ K \end{bmatrix};\quad \begin{bmatrix} i \\ j \\ k \end{bmatrix}=\mathbf{T}_{j_1}\begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix}=\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix}$$

(iii) $$\mathbf{T}=\begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}.$$

(b)(6pts) It can be shown that $\mathbf{T^{tr}T}=\mathbf{I}$. Since $\mathbf{T^{-1}}=\mathbf{T^{tr}}$, the rotation matrix **T** is said to be a *unitary* matrix. Since there are 9 terms to reckon with, here, show only that the (1,1) term is one, and that the (1,2) and (1,3) terms are zero.

Solution:

$$\mathbf{T}^{tr}\mathbf{T}=\begin{bmatrix} C\theta C\psi & -S\psi & S\theta C\psi \\ C\theta S\psi & C\psi & S\theta S\psi \\ -S\theta & 0 & C\theta \end{bmatrix}\begin{bmatrix} C\theta C\psi & C\theta S\psi & -S\theta \\ -S\psi & C\psi & 0 \\ S\theta C\psi & S\theta S\psi & C\theta \end{bmatrix}.$$ (1,1) term: $C\theta^2 C\psi^2+S\psi^2+S\theta^2 C\psi^2=C\psi^2+S\psi^2=1$.

(1,2) term: $C\theta^2 C\psi S\psi - C\psi S\psi + S\theta^2 C\psi S\psi = C\psi S\psi - C\psi S\psi = 0$.   (1,3) term: $C\theta S\theta C\psi - C\theta S\theta C\psi = 0$

(c)(4pts) Suppose that in the drone reference frame, its velocity is $\mathbf{v}=[u \quad v \quad w]^{tr}$, and the in the earth reference frame it is $\mathbf{V}=[V_x \quad V_y \quad V_z]^{tr}$. Then it should be clear that $\mathbf{v}=\mathbf{T}\,\mathbf{V}$, hence $\mathbf{V}=\mathbf{T}^{tr}\mathbf{v}$. Validate your answer for **T** in (a) by simplifying the matrix (call it **M**) in (3.30) on p.102 for $\phi=0$.

Solution: For $\phi=0$ we have $M=\begin{bmatrix} \cos\theta\cos\psi & -\sin\psi & \sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\psi & \sin\theta\sin\psi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$. This is exactly $\mathbf{T}^{tr}$.

PROBLEM 3(25pts) The linearized small disturbance longitudinal differential equations are given in Table 3.2 on p.108. It can be shown that $\dot{\theta} = q$. This set of equations can be expressed in the matrix form:

$$\begin{bmatrix}\Delta\dot{u} & \Delta\dot{w} & \Delta\dot{q} & \Delta\dot{\theta}\end{bmatrix}^{tr} = \mathbf{A}\begin{bmatrix}\Delta u & \Delta w & \Delta q & \Delta\theta\end{bmatrix}^{tr} + \mathbf{B}\begin{bmatrix}\delta_e & \delta_e\end{bmatrix}^{tr}.$$

(a)(20pts) Derive the expressions for **A** and **B** for the case where $Z_{\dot{w}} = 0$.

<u>Solution</u>:

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\end{bmatrix} = \begin{bmatrix} X_u\Delta u + X_w\Delta w - g\cos\theta_0\,\Delta\theta + X_{\delta_e}\delta_e + X_{\delta_T}\delta_T \\ Z_u\Delta u + Z_w\Delta w + (u_0 + Z_q)\Delta q + g\sin\theta_0\,\Delta\theta + Z_{\delta_e}\delta_e + Z_{\delta_T}\delta_T \\ M_u\Delta u + M_w\Delta w + M_{\dot{w}}\Delta\dot{w} + M_q\Delta q + M_{\delta_e}\delta_e + M_{\delta_T}\delta_T \end{bmatrix}. \ (1)$$

Substituting (1b) into (1c) gives:

$$\Delta\dot{q} = (M_u + M_{\dot{w}}Z_u)\Delta u + (M_w + M_{\dot{w}}Z_w)\Delta w + (M_q\Delta q + M_{\dot{w}}u_0)\Delta q + (M_{\delta_e} + M_{\dot{w}}Z_{\delta_e})\delta_e + (M_{\delta_T} + M_{\dot{w}}Z_{\delta_T})\delta_T. \ (2)$$

Substituting (2) into (1c) gives:

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g\cos\theta_0 \\ Z_u & Z_w & u_0 + Z_q & g\sin\theta_0 \\ M_u + M_{\dot{w}}Z_u & M_w + M_{\dot{w}}Z_w & M_q + M_{\dot{w}}u_0 & M_{\dot{w}}g\sin\theta_0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix} X_{\delta_e} & X_{\delta_T} \\ Z_{\delta_e} & Z_{\delta_T} \\ M_{\delta_e} + M_{\dot{w}}Z_{\delta_e} & M_{\delta_T} + M_{\dot{w}}Z_{\delta_T} \\ 0 & 0 \end{bmatrix}\begin{bmatrix}\delta_e\\ \delta_T\end{bmatrix}.$$

(b)(5pts) For $\theta_0 = 0$, $Z_q = M_{\dot{w}} = 0$, and $\delta_T = 0$ your answer in (a) should be

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix} X_{\delta_e}\\ Z_{\delta_e}\\ M_{\delta_e}\\ 0\end{bmatrix}\Delta\delta_e.$$ Edit the appropriate entries in your answer to verify it.

<u>Solution</u>:

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix} X_{\delta_e} & 0 \\ Z_{\delta_e} & 0 \\ M_{\delta_e} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix}\delta_e\\ 0\end{bmatrix}.$$ Verified ☺

PROBLEM 4(25pts) The longitudinal dynamics of a given plane are governed by $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$, where $\mathbf{x} = [\Delta u \quad \Delta w \quad \Delta q \quad \Delta\theta]^{tr}$, and where

$$\mathbf{A} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -.04 & .04 & 0 & -32.2 \\ -.4 & -2 & 180 & 0 \\ 0 & -.04 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We will soon investigate the properties of this dynamical system analytically. Here, we will use Matlab. To this end, we are required to have a *state equation* of the form $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\mathbf{u}$, and a *measurement equation* of the form $\mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\mathbf{u}$.

(a)(5pts) For our current problem $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$ it should be clear that the matrices **B** and **D** must be **0**, and that $\mathbf{C} = \mathbf{I}_{4\times 4}$ Suppose that $u = [u_1 \quad u_2]^{tr}$. Give the required dimensions for **B** and **D**.
<u>*Answer*</u>: $\mathbf{B} = \mathbf{0}_{4\times 2}$ and $\mathbf{D} = \mathbf{0}_{4\times 2}$.

(b)(10pts) Consider the initial condition $\mathbf{x}_0 = [20\, fps \quad 0 \quad \pi/6\, rad \quad 0]^{tr}$. Complete the Matlab code @ 4(b) to obtain plots of the initial condition responses. [NOTE: Units for $\Delta\theta(t)$ and $\Delta q(t)$ should be converted to degrees.]
<u>*Solution*</u>: [See code @ 4(b).]

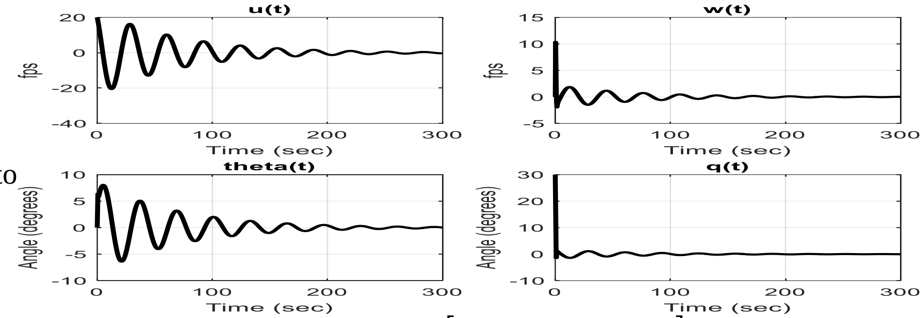


Figure 4(b) Responses $\mathbf{x} = [\Delta u \quad \Delta w \quad \Delta q \quad \Delta\theta]^{tr}$ over 300 sec.

(c)(5pts) Plot your results over the initial 10 seconds. To this end, simply copy/paste your code in (b) into a new part (c), and modify it.
<u>*Solution*</u>: [See code @ 4(c).]

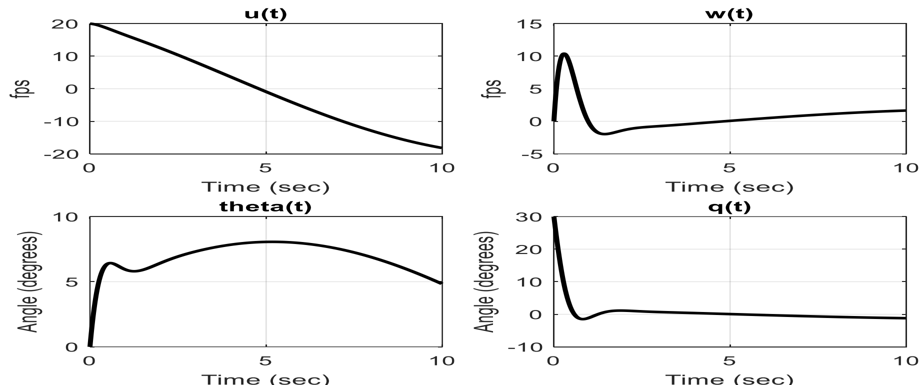


Figure 4(c) Responses $\mathbf{x} = [\Delta u \quad \Delta w \quad \Delta q \quad \Delta\theta]^{tr}$ over first 10 sec.

(d)(5pts) A comparison of your plots in (b) and (c) should reveal that, while $\Delta u(t)$ and $\Delta\theta(t)$ take a long time to die out, $\Delta w(t)$ and $\Delta q(t)$ essentially die out very quickly. These behaviors at two distinctly different time scales are reflective of two well-known modes of the longitudinal dynamics of a plane. Go to Wikipedia to arrive at their names and descriptions.
<u>*Answer*</u>: [https://en.wikipedia.org/wiki/Aircraft_dynamic_modes]

The longer period mode, called the "**phugoid mode**" is the one in which there is a large-amplitude variation of air-speed, pitch angle, and altitude, but almost no angle-of-attack variation. The phugoid oscillation is really a slow interchange of kinetic energy (velocity) and potential energy (height) about some equilibrium energy level as the aircraft attempts to re-establish the equilibrium level-flight condition from which it had been disturbed. The motion is so slow that the effects of inertia forces and damping forces are very low. Although the damping is very weak, the period is so long that the pilot usually corrects for this motion without being aware that the oscillation even exists. Typically the period is 20–60 seconds. This oscillation can generally be controlled by the pilot.

With no special name, the shorter period mode is called simply the "**short-period mode**". The short-period mode is a usually heavily damped oscillation with a period of only a few seconds. The motion is a rapid pitching of the aircraft about the center of gravity. The period is so short that the speed does not have time to change, so the oscillation is essentially an angle-of-attack variation. The time to damp the amplitude to one-half of its value is usually on the order of 1 second. Ability to quickly self damp when the stick is briefly displaced is one of the many criteria for general aircraft certification.

Appendix Matlab Code

```
%PROGRAM NAME: hw3.m
% PROBLEM 4
%(b):
A=[-.04 .04 0 -32.2;-.4 -2 180 0;0 -.04 -3 0;0 0 1 0];
C=eye(4); B=[]; D=[];
dt=.01; t=0:dt:300;
%----------------
du0=20; dq0=pi/6;
x0= [du0; 0; dq0 ; 0];
sys4 = ss(A,B,C,D);
g = initial(sys4,x0,t); %[du,dw,dq,dth]'
%---------------
figure(40)
subplot(2,2,1), plot(t,g(:,1),'k','LineWidth',2)
title('u(t)')
xlabel('Time (sec)')
ylabel('fps')
grid
hold on
subplot(2,2,3), plot(t,g(:,4)*(180/pi),'k','LineWidth',2)
title('theta(t)')
xlabel('Time (sec)')
ylabel('Angle (degrees)')
grid
subplot(2,2,2), plot(t,g(:,2),'k','LineWidth',2)
title('w(t)')
xlabel('Time (sec)')
ylabel('fps')
grid
subplot(2,2,4), plot(t,g(:,3)*180/pi,'k','LineWidth',2)
title('q(t)')
xlabel('Time (sec)')
xlabel('Time (sec)')
ylabel('Angle (degrees)')
grid
%----------------------------------------
%(c):
figure(41)
tmax=10;
n=tmax/dt;
tt=0:dt:tmax-dt;
subplot(2,2,1), plot(tt,g(1:n,1),'k','LineWidth',2)
title('u(t)')
xlabel('Time (sec)')
ylabel('fps')
grid
hold on
subplot(2,2,3), plot(tt,g(1:n,4)*(180/pi),'k','LineWidth',2)
title('theta(t)')
xlabel('Time (sec)')
ylabel('Angle (degrees)')
grid
subplot(2,2,2), plot(tt,g(1:n,2),'k','LineWidth',2)
title('w(t)')
xlabel('Time (sec)')
ylabel('fps')
grid
subplot(2,2,4), plot(tt,g(1:n,3)*180/pi,'k','LineWidth',2)
title('q(t)')
xlabel('Time (sec)')
xlabel('Time (sec)')
ylabel('Angle (degrees)')
grid
```