WEEK 10 LECTURES

I. Motivation

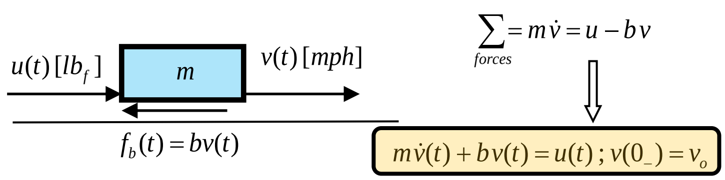
In view of the current cv19 situation, this set of lecture notes is intended to serve as not only a substitute for in-class lectures, but to provide the student with deeper insight into exam-related concepts. The intent is that this insight will address issues and provide answers to questions that students would otherwise ask in class during this week.

Rather than addressing such issues in a general framework, this set of notes will address them in the context of examples. The chosen examples will not be identical to problems included in the exam. Even so, they will be sufficiently similar so as to allows students to 'connect the dots'.

Finally, it should be noted that the examples will be addressed in all their 'gory' detail. The intent here is to anticipate questions or lack of understanding that even the sub-par student might be dealing with.

II. The Relation Between the Transfer Function and the Frequency Response Function

***Example* 1. A simple model for automobile velocity.**



This example was addressed in Lecture 1. The mass is being force to the right by a force $u(t)$ that we refer to as the 'input'. It imparts a velocity $v(t)$ to the mass. We refer to this velocity as the 'output. The retarding force $f_b(t) = bv(t)$ is a viscous friction force model. In other words, it is assumed that it is proportional to $v(t)$. A force balance yields the first order differential equation:

$$m\dot{v}(t) + bv(t) = u(t)\,. \qquad \text{(E1.1)}$$

Note that we have assumed the initial condition $v(0_-) = 0$.

To arrive at an expression for the output $v(t)$ in relation to any input $u(t)$, we will rely on

Definition 1. For any $x(t)$ defined over $0 \le t < \infty$, the **Laplace transform** of $x(t)$ is:

$$X(s) = \ell(x)(s) \overset{\Delta}{=} \int_{t=0}^{\infty} x(t)e^{-st}dt\,. \qquad \text{(D1)}$$

The variable s in (D1) is any chosen complex number $s = \sigma + i\omega$.

The most important property in relation to using (D1) to solve (E1.1) is the following:

$$X(s) = \ell(x^{(n)})(s) = s^n X(s)\,. \qquad \text{(P1)}$$

In words, what (P1) states is that the operation that is taking the nth derivative of $x(t)$ in the time domain is equivalent to simply multiplying $X(s)$ by s in the Laplace domain. Here again, it should be noted that (P1) has presumed any and all initial conditions that might be related to $x(t)$ are zero.

Applying (P1) in relation to (E1.1) gives:

$$msX(s) + bX(s) = U(s)\,. \tag{E1.2}$$

Before using (E1.2) to obtain the solution for $v(t)$, we offer the most important definition in the course.

Definition 2. For any input $u(t)$ and resulting output $y(t)$, the **system transfer function** is $G(s) \overset{\Delta}{=} \dfrac{Y(s)}{U(s)}$. (D2)

Applying (D2) in relation to (E1.2) gives:

$$G(s) = \frac{V(s)}{U(s)} = \frac{1}{ms+b}\,. \tag{E1.3}$$

The transfer function (E1.3) has a single pole: $s_1 = -b/m$. Since both m and b are greater than zero, this pole is negative (i.e. it is in the left half of the complex plane that we will denote as the LHP). Hence, (E1.3) is a stable system.

In order to obtain an expression for the solution $v(t)$, we need to be given a specific input. Here, we will assume that the input $u(t)$ is a unit step: $u(t) = u_s(t)$. We can now use a table of Laplace transform pairs to arrive at the solution $v(t)$. Specifically, we will use the following pairs:

$$u_s(t) \;\leftrightarrow\; U_s(s) = \frac{1}{s} \qquad \text{and} \qquad \left(\frac{1}{a}\right)\left(1-e^{-at}\right) \;\leftrightarrow\; \frac{1}{s(s+a)}\,. \tag{E1.4}$$

Now express (E1.3) as:

$$G(s) = \left(\frac{1}{m}\right)\frac{1}{s+(b/m)}\,. \tag{E1.5}$$

Then from (E1.3) we have:

$$V(s) = G(s)U(s) = \left[\left(\frac{1}{m}\right)\frac{1}{s+(b/m)}\right]\frac{1}{s} = \left(\frac{1}{m}\right)\left(\frac{m}{b}\right)\frac{1}{s[s+(b/m)]}\,. \tag{E1.6}$$

Applying the second table entry in (E1.4) to (E1.6) gives the solution:

$$v(t) = \left(\frac{1}{b}\right)\left(1 - e^{-(b/m)t}\right). \qquad \text{(E1.7)}$$

Numerical values- Let $m = 0.5$ and $b = 0.1$. Then (E1.7) becomes:

$$v(t) = 10\left(1 - e^{-t/5}\right). \qquad \text{(E1.8)}$$

The figure at right qualitatively verifies (E1.8). We see that the response achieves a steady state value $v_{ss} = 10$ at time $t = 25\,\text{sec}$. This time is equal to five 5τ, where the system time constant is $\tau = 5\,\text{sec}$. To be exact, the value of (E1.8) at $t = 25\,\text{sec}$. is $v(t = 25) = 9.9326$. In fact, mathematically, the model response (E1.8) will never achieve exactly 10.0. Then again, the viscous friction model, itself, is not exact. The goal is not to obtain an exact model. The goal is to obtain a model that captures the behavior of the system sufficient for our purposes.

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[m=0.5; b=0.1; G=tf(1,[m b]); figure(1) step(G)]
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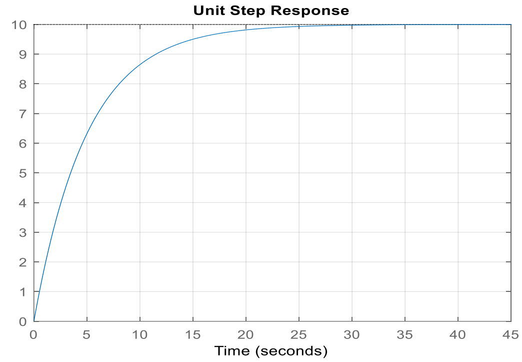

Figure E1.1 Step response for the TF (E1.5).

We will now proceed to address elements of this example that relate more directly to Exam 2.

Definition 3. The frequency response function (FRF) associated with a transfer function (TF) $G(s)$ is defined as $G(s = i\omega)$.

The FRF associated with (E1.5) is:

$$G(i\omega) = \frac{2}{0.2 + i\omega}. \qquad \text{(E1.9)}$$

The FRF (E1.9) is best expressed in polar (i.e. magnitude/angle) form:

$$G(i\omega) = \frac{2}{0.2 + i\omega} = \frac{2}{\sqrt{0.2^2 + \omega^2}} e^{i\left[-\tan^{-1}(\omega/0.2)\right]} \overset{\Delta}{=} M(\omega)e^{i\theta(\omega)}. \qquad \text{(E1.10)}$$

A plot of (E1.10) is shown below. It is called a Bode plot because of the log nature of the frequency axis and the magnitude. Specifically, the magnitude is $M_{dB} = 20\log_{10}(M)$.

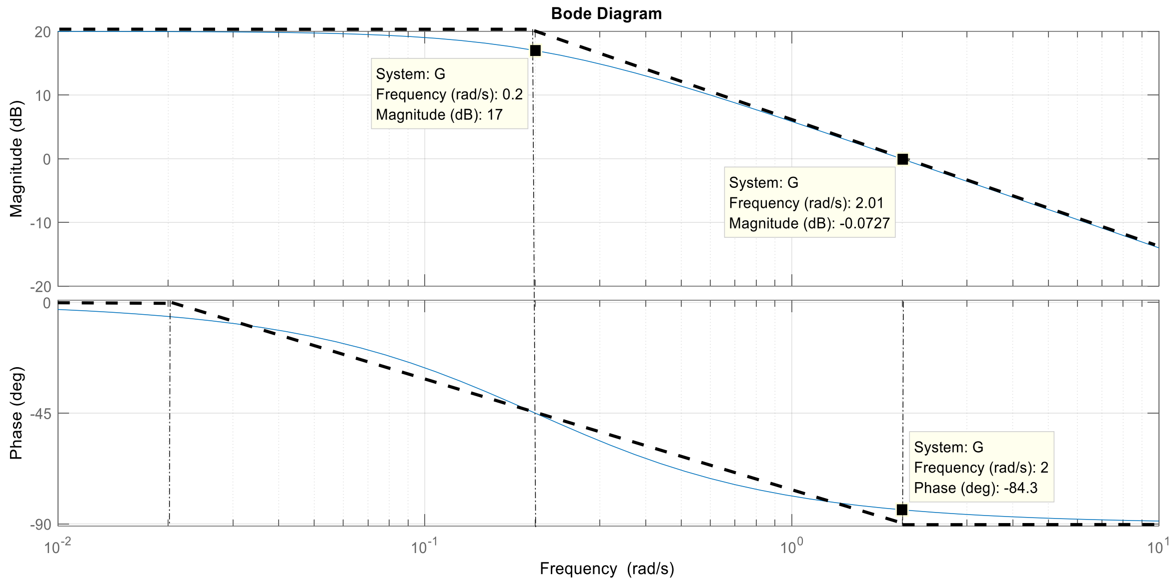

Figure E1.2 Bode plot associated with (E1.10).

This figure includes a notable amount of information. We will review it step-by-step.

(i) The dashed line information is called a straight-line Bode plot approximation. In relation to the magnitude M_{dB}, we see that the horizontal line changes to a line with a slope of $-20\,\text{dB/decade}$ at what is called the break frequency $\omega_{br} = 1/\tau = 2\,\text{rad/sec.}$ At very low frequencies $M_{dB} \cong 20\,\text{dB} = 20\log(g_s = 10)$, where the parameter $g_s = 10$ is the system *static* (i.e. low frequency) *gain*. The straight-line approximation is worst at the break frequency, where $M(\omega_{br})_{dB} = 20 - 3 = 17\,\text{dB}$. Because the magnitude is down 3dB from the static gain at ω_{br}, the frequency range $[0, \omega_{br}]$ is called the system -3dB *bandwidth*. It is the region where the FRF is within 3dB of its static gain.

In relation to the phase, from (E1.10) we have: $\theta(\omega) = -\tan^{-1}(\omega / 0.2)$. The phase straight-line approximation assumes that at frequencies below $0.1\omega_{br} = 0.2\,\text{rad/sec.}$ the phase is $\theta(\omega) \cong 0^o$, and at frequencies above $10\omega_{br} = 20\,\text{rad/sec.}$ it is $\theta(\omega) \cong -90^o$. At these two corner frequencies we actually have $\theta(0.2) = -5.7^o$ and $\theta(20) = -84.3^o$. We also see that the straight-line phase approximation is exact at ω_{br}: $\theta(\omega_{br}) = -45^o$.

(ii) Information in the other two data cursors relates to the placement of $G(s)$ into a feedback control configuration. In this context we will refer to $G(s)$ as the open loop TF. We see that $M(\omega_{gc} = 2)_{dB} = 0\text{dB}$. In other words, the gain crossover frequency $\omega_{gc} = 2\,rad/\text{sec}$. corresponds to $M(s = i\omega_{gc} = i2) = 1$. It is at this value of s that the root locus magnitude condition is satisfied. Since $s = i\omega_{gc}$ is on the imaginary axis, this frequency is of great concern. For, if we also had $\theta(s = i\omega_{gc} = i2) = -180^o$, then the root locus angle condition would also be satisfied. In this event, it would follow that $s = i\omega_{gc}$ is a closed loop (CL) system pole. Having a CL system pole on the imaginary axis corresponds to a marginally stable CL system.

Fortunately, from the second data cursor we have $\theta(s = i\omega_{gc} = i2) = -84.3^o$. This value is a full 95.7^o above the value -180^o. Hence, this value is called the CL system phase margin (PM).

Controller Design for Improved Performance

We will now address the design of a proportional controller $G_c(s) = K$ that achieves a $PM = 110^o$. To this end, we begin with a Bode plot of the OL system TF that includes the plant $G_p(s) = \frac{2}{s+0.2}$ and the controller $G_c(s) = 1$. The OL TF is then:

$$G(s) = G_c(s)G_p(s) = K\left(\frac{2}{s+0.2}\right) = \frac{2}{s+0.2}. \qquad \text{(E1.11)}$$

The Bode plot associated with (E1.11) is shown at right.

The lower data cursor shows $\theta(0.55) = -70^o$.

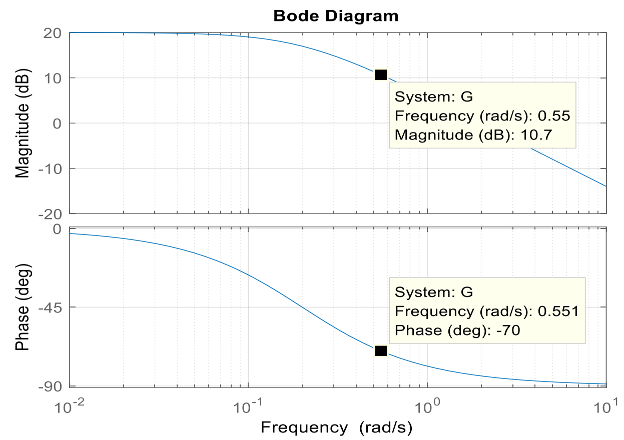

Figure E1.3. Bode plot of (E1.11).

This value is where the OL phase is 110^o above -180^o. Hence, if we can force the frequency $\omega = 0.55$ to be the gain crossover frequency, then we will have a CL system with $PM = 110^o$.

The upper data cursor shows that $M(0.55)_{dB} = 10.7$ dB. Hence, we will achieve $\omega_{gc} = 0.55$ rad/sec. for $K_{dB} = -10.7$ dB, or, equivalently, for $K = 10^{(-10.7/20)} = \mathbf{0.2917}$. The OL TF is now:

$$G(s) = \frac{2(0.2917)}{s+0.2}. \qquad \text{(E1.12)}$$

We see that we now have a CL $PM = 110^o$ We also see that the OL bandwidth decreased from the range $[0,2]$ to the range $[0,0.2]$.

The consequences of reducing the OL bandwidth by a factor of 10 are illustrated in the unity feedback CL FRF and unit step responses given below.

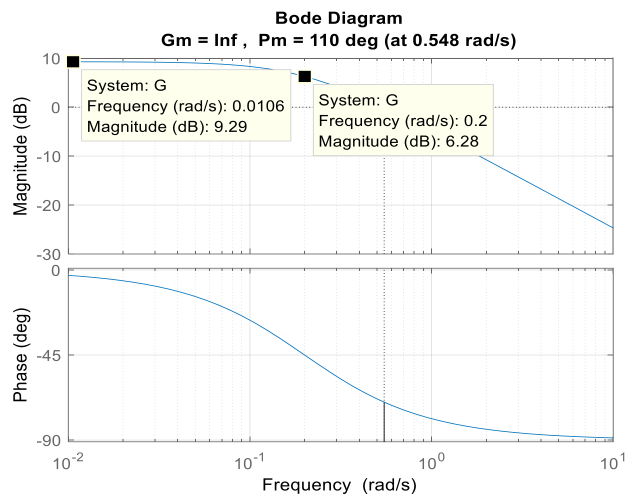

Figure E1.3. Bode plot of (E1.12).

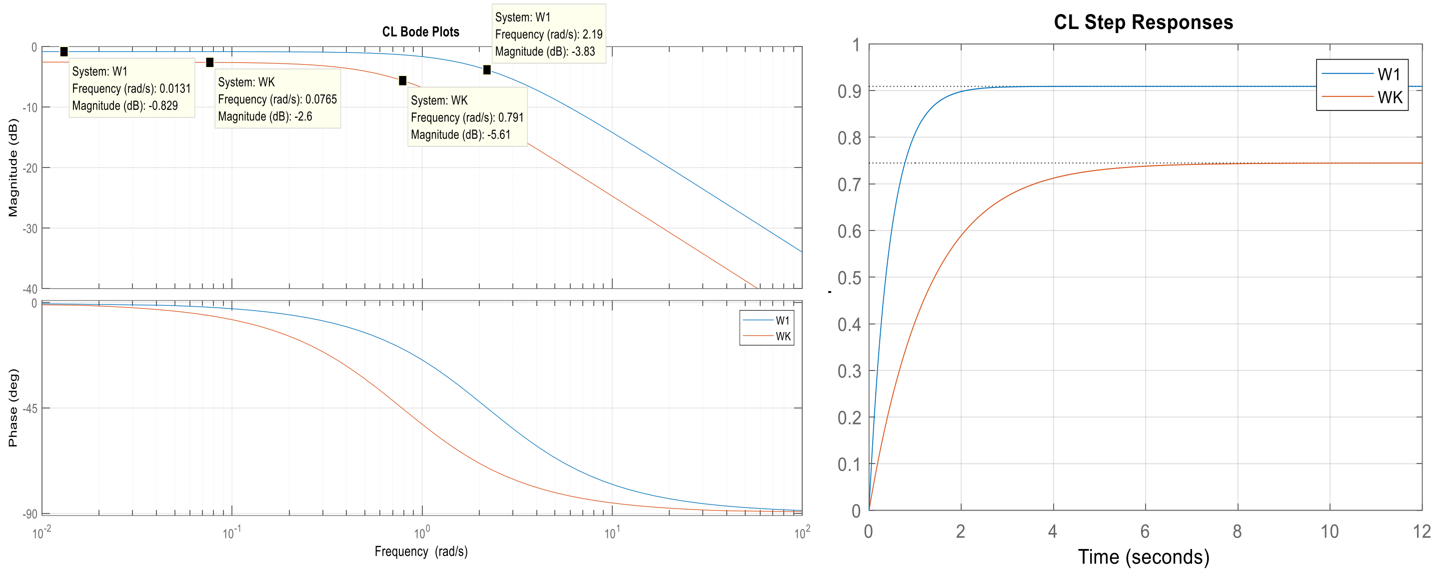

Figure (E1.4). CL Bode plots (LEFT) and step responses (RIGHT).

From the CL FRFs we see that by reducing the OL BW by a factor of 10, the CL BW was reduced by a factor of $2.19/0.79 \cong 2.78$. The larger the system BW, the more quickly it will respond. This BW reduction results in a notably slower step response, as shown at right.