The Q Factor of a Second Order System

Consider a second order underdamped system with transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \,. \qquad (1)$$

Notice the (1) was chosen to have unity static gain. This is merely a convenience.

Then $$P(\omega) = \frac{\omega_n^4}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2} = \frac{1}{(1-(\omega/\omega_n{}^2)^2 + 4\zeta^2(\omega/\omega_n)^2} = \frac{1}{(1-r)^2 + 4\zeta^2 r} \qquad (2)$$

where we have defined $r \overset{\Delta}{=} (\omega/\omega_n)^2$. $P(\omega)$ called the power spectrum associated with (1). It should be clear that $P(\omega)$ is the FRF magnitude-squared.

In order for (2) to have a resonance, there must be a value of *r* that makes the denominator less than 1. This, in turn will place a requirement on the range of ζ values. Setting $(1-r)^2 + 4\zeta^2 r < 1$ gives $r < 2(1-2\zeta^2)$. Since we cannot have $r < 0$, it follows that that we must have $\zeta < 1/\sqrt{2}$. This is worthy of

Result 1. A second order underdamped system will have no resonance for $\zeta \geq 1/\sqrt{2}$.

In view of this result, in the all of the following development, we will assume that $\zeta < 1/\sqrt{2}$.

We will now find the frequency, ω_{res} at which (2) achieves a maximum. Setting the derivative of (2) equal to zero and solving for r gives $r = 1 - 2\zeta^2$. Hence, (2) is maximum at the frequency:

$$\omega_{res} = \omega_n\sqrt{1-2\zeta^2} \,. \qquad (3a)$$

The value of (2) at the frequency (3a) is:

$$P(\omega_{res}) = \frac{1}{4\zeta^2(1-\zeta^2)} \,. \qquad (3b)$$

The frequency (3a) where the power of the FRF is a maximum is called the ***resonance*** (or resonant) ***frequency***.

The primary focus of these notes is on the Q-factor associated with (1).

Definition 1 The Q factor associated with an underdamped system is defined as $Q \overset{\Delta}{=} \omega_{res}/\Delta\omega$, where $\Delta\omega = \omega_2 - \omega_1$. The frequencies $\omega_{1,2}$ satisfy $P(\omega_{1,2}) = 0.5\,P(\omega_{res})$. They are called the *half-power* frequencies.

We will now arrive at the expressions for $\omega_{1,2}$. To this end, consider the frequency ω_0 satisfying $P(\omega_0) = 0.5\,P(\omega_{res})$. Then from (2) and (3b) we have:

$$\frac{1}{(1-r)^2+4\zeta^2 r}=\frac{0.5}{4\zeta^2(1-\zeta^2)} \tag{4}$$

where we have set $r \stackrel{\Delta}{=} (\omega_0/\omega_n)^2$. From (4) we have:

$$(1-r)^2+4\zeta^2 r=8\zeta^2(1-\zeta^2)\,. \tag{5}$$

This, in turn, gives:

$$r^2-2(1-2\zeta^2)r+[1-8\zeta^2(1-\zeta^2)]=0\,. \tag{6}$$

The solution to (6) is:

$$r_{1,2}=(1-2\zeta^2)\pm 2\zeta\sqrt{1-\zeta^2}\,. \tag{7}$$

From (7) we arrive at:

$$\omega_0^2=\omega_n^2\left[(1-2\zeta^2)\pm 2\zeta\sqrt{1-\zeta^2}\right]. \tag{8}$$

Using (3), (8) becomes:

$$\omega_0^2=\frac{\omega_{res}^2}{1-2\zeta^2}\left[(1-2\zeta^2)\pm\sqrt{1-8\zeta^2(1-\zeta^2)}\right]. \tag{9}$$

Hence, we arrive at:

$$\omega_{1,2}=\omega_{res}\sqrt{1\pm\frac{2\zeta}{1-2\zeta^2}\sqrt{1-\zeta^2}}\,. \tag{10}$$

Before we go further, it should be pointed out that in (10) we must have $\omega_1>0$. It is easy to show that this requires that

$$\zeta\le 0.38265\,. \tag{11}$$

Hence, we see that it is (11) and not $\zeta<1/\sqrt{2}$ that bounds ζ.

From (7) we have:

$$\sqrt{r_{1,2}}=\sqrt{1\pm\frac{2\zeta}{1-2\zeta^2}\sqrt{1-\zeta^2}}\,. \tag{12}$$

We then have

Result 2. The Q-factor associated with (1) is: $Q=1/(\sqrt{r_2}-\sqrt{r_1})$ for $\zeta\le 0.38265$. For $\zeta>0.38265$, Q is not defined.

Remark 1. Many textbooks on the subject give the approximate expression $\hat{Q} = 1/2\zeta$. The accuracy of this approximation is shown at right.

We see that, indeed, $\hat{Q} = 1/2\zeta$ is an excellent approximation of $Q = 1/(c_2 - c_1)$ for $\zeta \le 0.2$. For $\zeta = 0.38265$, $\hat{Q}$ is ~2.6dB higher than the true value. Many textbooks fail to mention is that it is valid for only $\zeta \le 0.38265$. Moreover, they do not give the expression for the true *Q*. Fortunately, *Q* is rarely addressed for damping ratios $\zeta > 0.25$.

[Note: $Q_{dB} = 20\log Q$.]

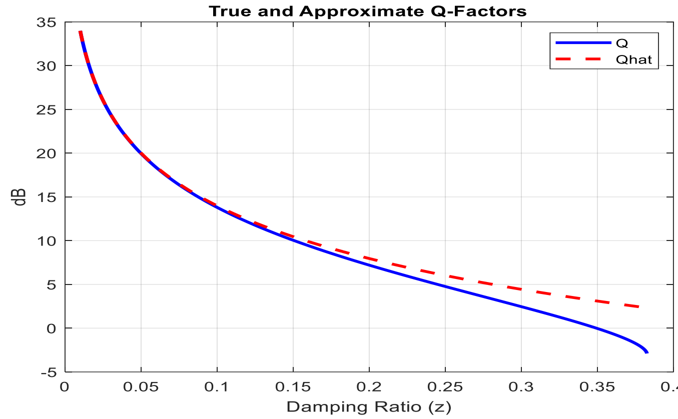

Figure 1 Plots of $Q = 1/(c_2 - c_1)$ and $\hat{Q} = 1/2\zeta$.

The FRF plots below offer some visual appreciation.

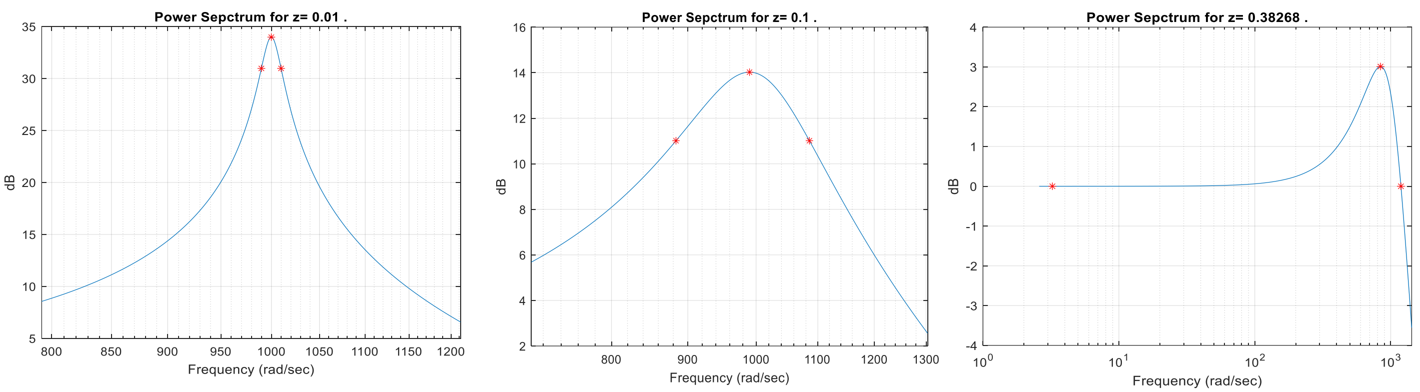

Figure 2. Power Spectrum plots for $\zeta = 0.01$(LEFT), $\zeta = 0.1$(CENTER) and $\zeta = 0.38268$(RIGHT).

A Brief Look Into the More General Case with Poles at the Origin

We will now consider the more general case where

$$G(s) = \frac{\omega_n^{2+m}}{s^m(s^2 + 2\zeta\omega_n s + \omega_n^2)}. \tag{13}$$

Note that the power $2+m$ in the numerator of (13) is, again, merely for convenience. Because (13) has *m* poles at the origin, its low frequency power behavior will entail a slope of $-20m$ dB/dec.

One should suspect that this roll-off will reduce the range of ζ-values such that the peak at resonance is at least 3dB above the value at ω_1. As was done in relation to (2), let $r \overset{\Delta}{=} (\omega/\omega_n)^2$. Then the power spectrum associated with (13) is:

$$P(r) = \frac{1}{r^m[(1-r)^2 + 4\zeta^2 r]} \tag{14}$$

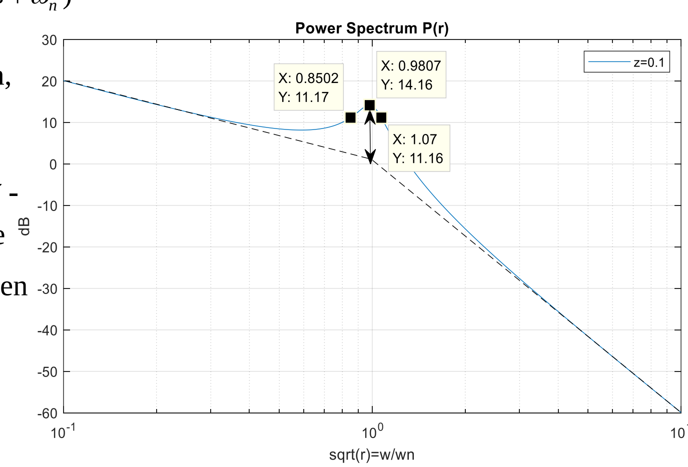

Figure 3. The power spectrum (14) for $\zeta = 0.1$.

Figure 3 shows the power spectrum (14) for $\zeta = 0.1$. The -3dB relative BW is $\Delta r = r_2 - r_1 = 1.07 - 0.85 = 0.22$, and the relative resonance frequency is $r_{res} = 0.98$. Hence, $Q = r_{res} / \Delta r = 4.45$, or $Q_{dB} = 12.97\,\text{dB}$. This value is indicated by the double arrow in Figure 3. If we approximate Q by $\hat{Q} = 1/2\zeta = 5$, or $\hat{Q}_{dB} = 13.98\,\text{dB}$, we see that it is about 2dB higher than the true value. It is important to note that the double arrow was obtained by first constructing a straight-line approximation of the power spectrum.

Note from Figure 3 that $P(r)$ has both a relative minimum and a relative maximum. Setting $P'(r) = 0$ gives:

$$c_{1,2}^{(m)} = \left[(m+1)(1-2\zeta^2) \pm \sqrt{1-4(m+1)^2\zeta^2(1-\zeta^2)} \right] / (m+2) \,. \tag{15}$$

The square root of the smaller element of (15) gives the relative *minimum* scaled frequency prior to the resonance, while the larger element gives the *resonance* scaled frequency.

Proceeding to use (15) in order to arrive at an explicit expression for $P(r_{res})$ would allow us to identify the largest value for ζ such that the Q-factor is defined. Instead, we simply searched for it. The result of this search is shown in Figure 4.

The relative frequencies computed from (15) are $\sqrt{c_1^{(1)}} = 0.594$ and $\sqrt{c_2^{(1)}} = \sqrt{r_{res}^{(1)}} = 0.945$. They match well with 0.6034 and 0.9564, respectively. The damping ratio $\zeta = 0.1475$ results in a 3dB dip between the two frequencies. Hence, $\zeta = 0.1475$ is the largest damping ratio for which a *Q*-factor is defined for $m = 1$. The true and estimated Q-factors are $Q = 0.9564/(1.086 - .6034) = 1.98 = 5.94dB$ and $\hat{Q} = 1/(2\times 0.145) = 3.39 = 10.6dB$. We see that the estimate is 6.5dB greater than the truth.

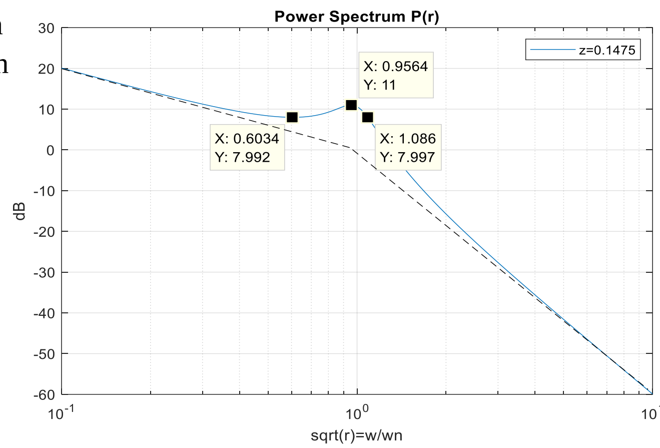

Figure 4. Power spectrum for maximum $\zeta = 0.1475$.

While it would have been instructive to arrive at the expression for Q by finding expressions for the -3dB frequencies $\omega_1^{(1)}$ and $\omega_2^{(1)}$ as was done for $m = 0$, we did, at least find the expression for the resonance frequency $\omega_{res}^{(m)} = \omega_n \sqrt{c_2^{(m)}}$.

Result 2. For $m = 1$, the *Q*-factor is not defined for $\zeta \gtrsim 0.1475$. The resonance frequency for any *m* is $\omega_{res}^{(m)} = \omega_n \sqrt{c_2^{(m)}}$.

Conclusions: For second order underdamped systems, the *Q*-factor is commonly taken to be $\hat{Q} = 1/2\zeta$. In fact, in some textbooks it is taken as the definition of the *Q*-factor. If one accepts Definition 1, then:

- For $m = 0$ $\hat{Q} = 1/2\zeta$ is a reasonable estimate of Q for $\zeta \lesssim 0.25$. A *Q*-factor only exists for $\zeta \lesssim 0.38$.

- For $m = 1$ $\hat{Q} = 1/2\zeta$ is a reasonable estimate of Q for $\zeta \lesssim 0.1$. A *Q*-factor only exists for $\zeta \lesssim 0.14$.

- For $m = 2$ one can expect that A *Q*-factor only exists for, at most, $\zeta \lesssim 0.1$.

- The above bullets are perhaps the likely reason that often engineers do not address a Q-factor for $\zeta \gtrsim 0.1$.
  I, myself am guilty of having violated this bound ☹

Matlab Code

```
%PROGRAM NAME: Qfactors.m
%This code investigates two definitions of the Q-factor
%for a 2nd order system:
%Qhat=(2*z)^-1
%Q=wr/dw where dw is the 1/2 power BW range.
%---------------------------------------------
z=0.01:.0001:.38265;
%max z for r1>0 is 0.3826
r1=sqrt(1-2*z.*(1-2*z.^2).^-1.*sqrt(1-z.^2));
sqr=sqrt(1+2*z.*(1-2*z.^2).^-1.*sqrt(1-z.^2));
Q=(sqr-r1).^-1;
Qhat=0.5*z.^-1;
QdB=20*log10(Q);
QhatdB=20*log10(Qhat);
figure(1)
plot(z,QdB,'b','LineWidth',2)
hold on
plot(z,QhatdB,'r--','LineWidth',2)
title('True and Approximate Q-Factors')
legend('Q','Qhat')
ylabel('dB')
xlabel('Damping Ratio (z)')
grid
wn=1000;
z=0.38268;
wr=wn*sqrt(1-2*z^2);
Pwr=1/(4*z^2*(1-z^2));
PwrdB=10*log10(Pwr);
Q1=(2*z)^-1
%----------------
w1=wr*sqrt(1-2*z*(1-2*z^2)^-1*sqrt(1-z^2));
w2=wr*sqrt(1+2*z*(1-2*z^2)^-1*sqrt(1-z^2));
Pwr1=Pwr/2; Pwr1dB=10*log10(Pwr1);
Pwr2=Pwr1;  Pwr2dB=Pwr1dB;
dw=w2-w1;
Q2=wr/dw
%---------------------
w=0.8*w1:.0001:1.2*w2;
M2=wn^4*((wn^2-w.^2).^2+4*z^2*wn^2*w.^2).^-1;
M2dB=10*log10(M2);
figure(2)
semilogx(w,M2dB)
grid
hold on
plot([wr w1 w2],[PwrdB Pwr1dB Pwr2dB],'*r')
title(['Power Sepctrum for z= ',num2str(z),' .'])
xlabel('Frequency (rad/sec)')
ylabel('dB')
%===========================================
%Include m poles @ the origin:
m=1;
z=0.1475;
r=0.01:.001:100;
P=(r.^m.*((1-r).^2+4*z^2*r)).^-1;
PdB=10*log10(P);
sqr=sqrt(r);
figure(11)
semilogx(sqr,PdB)
title('Power Spectrum P(r)')
xlabel('sqrt(r)=w/wn')
ylabel('dB')
legend('z=0.1475')
grid
%---------------------------------
% -3dB relative frequencies:
rctr=((m+1)/(m+2))*1-2*z^2;
dr=sqrt(1-(m+1)^2*4*z^2*(1-z^2))/(m+2);
r1=rctr-dr;
rres=rctr+dr;
sqr1=sqrt(r1);
sqrres=sqrt(rres);
[sqr1 sqrres]
```