LECTURE 9 **Review of Material to Date**

1. Dynamical Systems

A dynamical system is a differential equation that describes how various system outputs are related to various inputs and initial conditions. In this course we restrict our attention to ordinary differential equations having constant coefficients. The general form of such an equation is:

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t) = b_m f^{(m)}(t) + b_{m-1} f^{(m-1)}(t) + \cdots + b_1 f^{(1)}(t) + b_0 f(t) . \qquad (1)$$

Complete specification of (1) requires specification of the initial conditions $\{y^{(k)}(0\_)\}_{k=0}^{n-1}$. To solve such an equation we appealed to the standard method of first considering the homogeneous equation:

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t) = 0 . \qquad (2)$$

The solution of (2), which is called the complementary solution, must have the form

$$y(t) = Ce^{st} . \qquad (3)$$

It follows that

$$y^{(k)}(t) = Cs^k e^{st} . \qquad (4)$$

Substituting (4) into (2) gives:

$$C[a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0] e^{st} = 0 . \qquad (5)$$

The trivial solution of (5) is letting $C = 0$. Since the term $e^{st} \neq 0$ for any *t*, the nontrivial solution of (5) must satisfy

$$p(s) \stackrel{\Delta}{=} a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 . \qquad (6)$$

The n^{th} degree polynomial $p(s)$ defined in (6) has n roots $\{s_k\}_{k=1}^{n-1}$.

We then proceeded to carry out the complete solution of (1) via the standard method of obtaining the particular solution for a given input $f(t)$, and then solving for the coefficients $\{C_k\}_{k=0}^{n-1}$ based on the given $\{y^{(k)}(0\_)\}_{k=0}^{n-1}$.

We then proceeded to offer an alternative to the above method, based on the notion of a Laplace transform:

$$\ell[x](s) = \int_{t=0}^{\infty} x(t) e^{-st} dt \stackrel{\Delta}{=} X(s) \cdot \qquad (7)$$

In order to apply (7) to obtain the complete solution of (1) under non-zero initial conditions, we derived the following result:

$$\ell(y^{(1)})(s) = sY(s) - y(0\_) . \qquad (8)$$

It was then pointed out that from (7), we obtain $\ell(y^{(2)})(s) = s[sY(s) - y(0\_)] - y^{(1)}(0\_)$, and that we could keep on applying this recursively for $\ell(y^{(k)})(s)$. Doing so, leads to the use of simple algebra to solve for $Y(s)$, from which $y(t)$ is obtained through use of a table of Laplace transform pairs.

2. Transfer Functions

The transfer function $G(s)$ associated with (1) is obtained by taking its Laplace transform under zero initial conditions:

$$G(s) \overset{\Delta}{=} \frac{Y(s)}{F(s)} = \frac{b_m s^m + b_{m-1}s^{(m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \overset{\Delta}{=} \frac{z(s)}{p(s)}. \tag{9}$$

Notice that the units of $G(s)$ are the units of $y(t)$ divided by those of $f(t)$. The transfer function ***poles*** are the values of s that result in $G(s) = \infty$, and the transfer function ***zeros*** are the values of s that result in $G(s) = 0$. Comparing (9) and (6), we see that the poles are the roots of the characteristic polynomial. Express the k^th^ pole as $s_k = \sigma_k + i\omega_k$. Then the assumed solution (3) for that pole is:

$$y_k(t) = C_k e^{s_k t} = C_k e^{\sigma_k t} e^{i\omega_k t}. \tag{10}$$

The term $e^{i\omega_k t} = \cos(\omega_k t) + i\sin(\omega_k t)$ has unit magnitude for all *t*. It follows that:

$$\text{(C1):}\ \lim_{t\to\infty} y_k(t) = 0 \text{ for } \sigma_k < 0. \qquad \text{(C2):}\ \lim_{t\to\infty} y_k(t) = \infty \text{ for } \sigma_k > 0. \tag{11}$$

The conditions (C1) and (C2) reflect stable and unstable behavior, respectively. Hence, we have the following result.

Result 1 A system is stable only if all the poles of $G(s)$ are in the left half of the complex plane.

A word is now in order in relation to $\ell[*](s)$ defined in (7). It is often claimed that $\ell[*](s)$ is a linear operator; that is: $\ell[ax+by](s) = a\ell(x)(s) + b\ell(y)(s) = aX(s) + bY(s)$. While this is true if all initial conditions are assumed to be zero, it is not true in general.

The beauty of (9) is that, given $G(s)$ and $F(s)$, then

$$Y(s) = G(s)F(s). \tag{12}$$

Hence, under zero initial conditions the solution of (1) that is, in the Laplace domain (12), is trivial. The only effort needed is to use a table of Laplace transform pairs to arrive at $y(t)$

3. Block Diagrams and Closed Loop Systems

Consider the feedback control system block diagram shown at right. From (12) it should be clear that the output of any block is simply the transfer function times the input to that block.

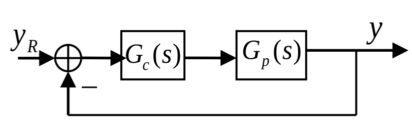


Figure 1 Unity feedback block diagram.

Hence, using simple algebra in the Laplace domain we arrive at the closed loop system transfer function

$$W(s) = \frac{Y(s)}{Y_R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}. \tag{13}$$

We start with the uncontrolled system $G_p(s)$ that is called the ***plant***. If we are given specifications in relation to $W(s)$, (13) allows us to design a controller $G_c(s)$ to achieve them.

4. Proportional + Integral + Derivative (PID) Control

The controller

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s \tag{14}$$

Is called a PID controller. It includes three parameters that permit us to design a controller that will achieve given closed loop specifications. It can also be expressed as:

$$G_c(s) = \frac{K(s-z_1)(s-z_2)}{s}. \tag{15}$$

The form (15) makes it clear that a PID controller has two zeros and one pole. The pole is at $s=0$, and is due to the integral control term.

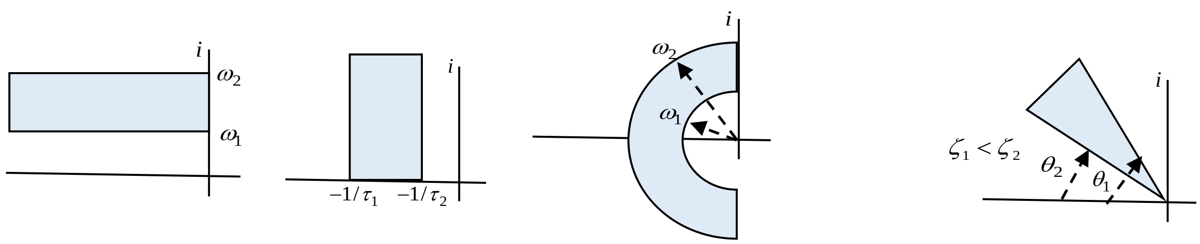
5. The Pole-Placement Method of Controller Design

Before describing this method it is crucial that the student have a solid understanding of complex poles and their relation to various specifications. While it was presented in Lecture 6, the following bears repeating.

Consider the polynomial $p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$ having complex-conjugate roots. From the quadratic formula, the roots of $p(s)$ are:

$$s_{1.2} = -\zeta\omega_n \pm i\,\omega_d \quad where \quad \omega_d \overset{\Delta}{=} \omega_n\sqrt{1-\zeta^2} \ \text{ for } \ 0 \le \zeta < 1. \tag{16}$$

The plots below illustrate the geometry associated with (1).



(i) $\omega_1 \le \omega_d \le \omega_2$ **(ii)** $\tau_1 \le \tau \le \tau_2$ **(iii)** $\omega_1 \le \omega_n \le \omega_2$ **(iv)** $\zeta_1 \le \zeta \le \zeta_2$

Figure 2 The plots show areas of constant (i) ω_d, (ii) τ, (iii) ζ, and (iv) ω_n.

Figure 2 illustrates the relation between the system pole $s_1 = -\zeta\omega_n + i\,\omega_d$ and specifications involving τ, ζ, ω_d, and ω_n.

The pole-placement method is one where the closed loop poles are specified (or placed), and the root locus angle and magnitude criteria are used to design the needed controller. We will now review the development of these criteria.

From (13) we see that the closed loop poles are the values of s that make $1 + G_c(s)G_p(s) = 0$; or, equivalently, that make

$$G_c(s)G_p(s) = -1 = 1e^{-i\pi}. \tag{17}$$

The reason for the rightmost equality in (17) will now become clear. Write

$$G_c(s)G_p(s) = G(s) = K\frac{\prod_{k=1}^{m}(s-z_k)}{\prod_{k=1}^{n}(s-p_k)}. \tag{18}$$

Now consider the term $s - z_k$. This is a difference vector with its tail at z_k and its tip at *s*. It has a magnitude $|s - z_k|$ and an angle ϕ_k^z (measured ccw from the + real axis) that points from z_k to *s*. In polar form, $s - z_k = |s - z_k| e^{i\phi_k^z}$. Hence, from (17) and (18):

$$G_c(s)G_p(s) = G(s) = K\frac{\prod_{k=1}^{m}|s-z_k|}{\prod_{k=1}^{n}|s-p_k|} e^{i\left(\sum_{k=1}^{m}\phi_k^z - \sum_{k=1}^{n}\phi_k^p\right)} = 1e^{-i\pi} \tag{18}$$

From (18) we see that s will be a closed loop pole if the following conditions are satisfied:

The angle condition: $\sum_{k=1}^{m}\phi_k^z - \sum_{k=1}^{n}\phi_k^p = -\pi$. ***The magnitude condition***: $K = \frac{\prod_{k=1}^{n}|s-p_k|}{\prod_{k=1}^{m}|s-z_k|}$. (19)

We see that the angle condition will determine if the specified s is a pole of $W(s)$ for some gain *K*, and that the magnitude condition will give the required value for *K*.

As an example of pole-placement PID controller design, one would first compute

$$\sum_{k=1}^{m}\phi_k^z - \sum_{k=1}^{n}\phi_k^p = \phi \tag{20$$

in relation to $G_p(s)/s$. The angle $\phi + \pi \overset{\Delta}{=} \theta$ is the amount of angle that the two controller zeros need to add in order to satisfy the angle criterion. I term the angle $-\theta$ the *defect angle*. Finally, having determined values for the controller zeros $z_{1,2}$, the magnitude criterion is used to find the value for *K*.

It should be noted that *K* is not necessarily the controller *K*. For, if the coefficients of the highest power of *s* in the numerator or denominator of $G_p(s)/s$ is not unity, then the *K* in the magnitude criterion (19) must include it.

6. The Frequency Response Function (FRF)

For a stable system with transfer function $Y(s)/F(s) = G(s)$, the FRF is $G(i\omega)$, which can be expressed in polar form as $G(i\omega) = M(\omega)e^{i\theta(\omega)}$. For input $f(t) = A\sin(\omega t + \phi)$, the ***steady state*** output is $y(t) = [M(\omega)A]\sin[\omega t + \phi + \theta(\omega)]$. Hence, the FRF does two things to a sinusoidal input: (i) It scales the magnitude by $M(\omega)$, and (ii) it shifts the phase by $\theta(\omega)$.

The importance of the FRF in relation to a plant-
Here, by the term plant we mean the system of primary concern. The FRF describes the plant dynamics in the frequency domain. For example, a major peak in the FRF reflects a system resonance.

The importance of the FRF in relation to a sensor-
While a sensor is, itself, a dynamical system, it is usually not the primary system of concern. The output of many sensors is voltage. If the sensor is designed to measure a physical variable (e.g. position, velocity, etc.), then it is desirable that there be a single scale factor relating that variable to voltage at every frequency of interest. In other words, it is desirable

that the sensor FRF magnitude be flat over the frequency range of interest. The useful frequency range of a given sensor is typically defined as that range where its FRF magnitude is within +/- 3 dB of being flat.

The importance of the FRF in relation to a controller-

The total power associated with a controller with transfer function $G_c(s)$ is $pwr = \frac{1}{2\pi}\int_{-\infty}^{\infty}|G_c(i\omega)|^2\,d\omega = \frac{1}{\pi}\int_{0}^{\infty}|G_c(i\omega)|^2\,d\omega$.

Most often the FRF magnitude is expressed in dB: $M(\omega)_{dB} = 20\log_{10} M(\omega)$. Equivalently, one can define the FRF power as $P(\omega)_{dB} = 10\log_{10} M(\omega)^2$. Since $P(\omega)_{dB} = 10\log_{10} M(\omega)^2 = 20\log_{10} M(\omega) = M(\omega)_{dB}$, the dB units on the vertical axis of the FRF plot can be viewed as power or magnitude. In the case of the former one uses the multiplier 10, whereas in the case of the latter one uses the multiplier 20.