Lecture 7 The Mathematics of the Root Locus

Example 1. Consider a feedback control system with open loop $G(s) = K\frac{s+2}{s(s+5)} = K\frac{s-z_1}{(s-p_1)(s-p_2)}$. The closed loop poles are the values of s that solve the equation: $1+K\frac{s-z_1}{(s-p_1)(s-p_2)} = 0$. This equation is exactly the equation:

$$K\frac{s-z_1}{(s-p_1)(s-p_2)} = \frac{K\,|s-z_1|e^{i\phi_{z_1}}}{|s-p_1|e^{i\phi_{p_1}}\,|s-p_2|e^{i\phi_{p_2}}} = 1e^{\pm i\pi}\ . \qquad (1)$$

The polar form in (1) is really helpful, since this single equation can be replaced by the following two equations:

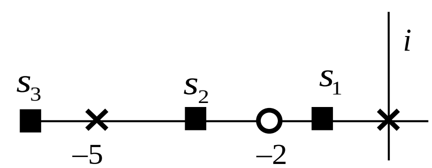
$$\frac{K\,|s-z_1|}{|s-p_1|\,|s-p_2|} = 1 \quad \text{(2a)} \qquad\qquad \phi_{z_1} - (\phi_{p_1}+\phi_{p_2}) = \pm\pi \quad \text{(2b)} \qquad\qquad (2)$$

Hence, s will be a closed loop pole if and only if it satisfies the left equation (the RL *magnitude condition*) and the right equation (the RL *angle condition*).

The magnitude condition allows one to compute the required value of K to solve it for almost <u>any</u> chosen value of s. Once we have found the value for z_1 that satisfies (2b), only then does (2a) permit us to compute what K must be.

It is the angle condition that governs the behavior of the root locus as $K = 0 \to \infty$. Consequently, the key to understanding the behavior of the root locus is to understand the *angles* involved in (2b).

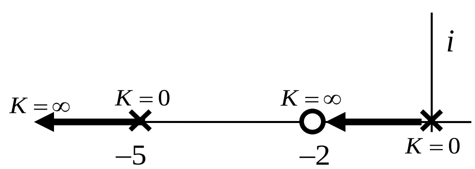
The geometry associated with (2b) is shown at right for three closed loop pole candidates. Let's now **prove** that both s_1 and s_3 are viable candidates, while s_2 is not.



$s_1:\ \varphi_{z_1} - (\varphi_{p_1}+\varphi_{p_2}) = 0-(\pi+0) = \pi$; $s_2:\ \varphi_{z_1} - (\varphi_{p_1}+\varphi_{p_2}) = \pi-(\pi+0) = 0$;
$s_3:\ \varphi_{z_1} - (\varphi_{p_1}+\varphi_{p_2}) = \pi-(\pi+\pi) = -\pi$.

What we have proven here is a special case of the more general result concerning real-valued closed loop pole (i.e. root) candidates:

***Result* 1.** The root locus will include every region on the real axis that is to the left of an odd number of open loop poles and/or zeros.



This result is illustrated at right. Notice also that each locus begins at an open loop pole and ends at an open loop zero. The plot shows only one finite zero, and so there must be another at $-\infty$.

Let's place a closed loop pole at $s=-0.5$. Then the magnitude criterion (2a) gives:

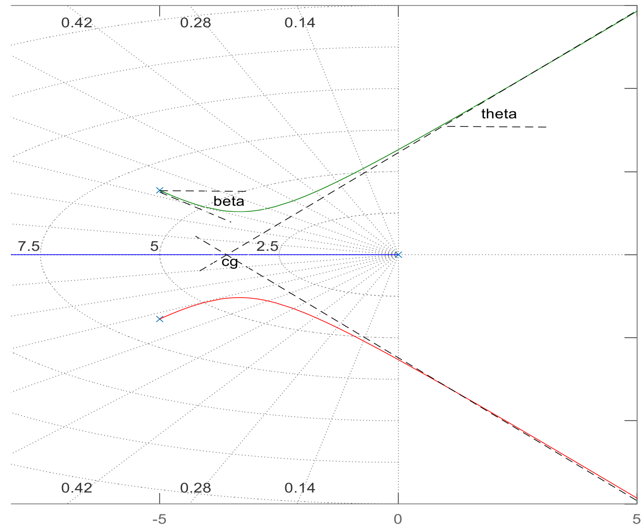
$\frac{K\,|s-z_1|}{|s-p_1|\,|s-p_2|} = \frac{K(1.5)}{(0.5)(4.5)} = 1$. Hence, we require $K = 1.5$ to place this pole at the desired location.

QUESTION: Can you now figure out where the second closed loop pole is? □

The general root locus criteria are:

$$K\frac{\prod_{k=1}^{m}|s-z_k|}{\prod_{k=1}^{n}|s-p_k|}=1 \qquad (3a) \quad ; \qquad \sum_{k=1}^{m}\varphi_{z_k}-\sum_{k=1}^{m}\varphi_{p_k}=\pm 180^o \qquad (3b) \qquad\qquad (3)$$

Example 2. For $G(s)=\dfrac{25}{s(s^2+2s+25)}$ the CL root locus is at right.

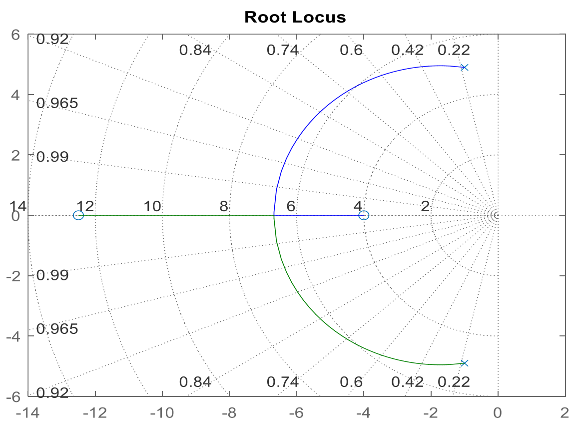


(i)Find theta: IN-CLASS

(ii)Find beta: IN-CLASS

(iii)Find cg: IN-CLASS

Example 3. Consider $G_p(s)=\dfrac{2s+25}{s^2+2s+25}$ and $G_c(s)=s+4$.
Then $G_p(s)=\dfrac{2(s+12.5)(s+4)}{s^2+2s+25}$. The CL root locus is at right.



(i)Use the angle criterion to find the departure angle from the complex poles.

IN-CLASS

(i)Use the gain expression to find the point at which the locus intersects the real axis.

IN-CLASS

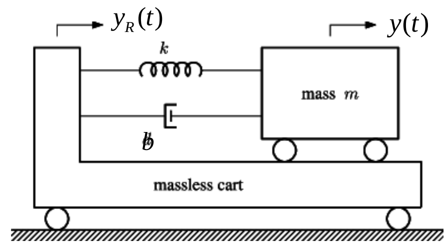
Remark 1. The book gives only 5 'RULES' for root locus plotting. This is better than in the 1^{st} edition; which gave 12.

Remark 2. The book goes into detail in *Design Using Lead Compensation* (5.4.1). This is important reading. We will also cover *Lag Compensation* and *Lead/Lag Compensation*. So please read this section.

Remark 3. Example 5.13 addresses the design of an autopilot for a Piper Dakota aircraft. This is an excellent example of how complicated things can be in the real world. Please read through this example.

Example 4. [Another real design problem- IN CLASS ☺] In this example we have the plant $G_p(s) = \dfrac{2s+25}{s^2+2s+25}$.

Q1: What might this plant correspond to?

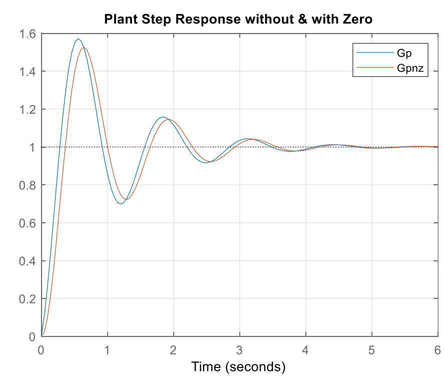


$\sum f = m\ddot{y} = b(\dot{y}_R - \dot{y}) + k(y_R - y)$ gives $m\ddot{y} + b\dot{y} + ky = b\dot{y}_R + ky_R$. Hence:

$$\frac{Y(s)}{Y_R(s)} = G_p(s) = \frac{bs+k}{ms^2+bs+k} = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with poles at $p_{1,2} = \zeta\omega_n \pm i\omega_d$ and with a zero at $z_1 = -\omega_n / 2\zeta$. For $G_p(s) = \dfrac{2s+25}{s^2+2s+25}$: $p_{1,2} = 1 \pm i\,4.9$ and $z_1 = -12.5$.

Q2: What is behavior of the plant that might warrant feedback control?



As seen in the plot at right, the response has significant overshoot. Furthermore, it is dictated mainly from the poles. It also has a settling time of ~5 sec.

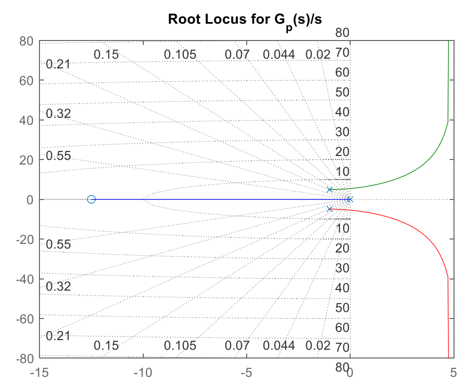
Q3: What might be the desired behavior?

Faster response, minimal overshoot, and accurate steady state positioning.

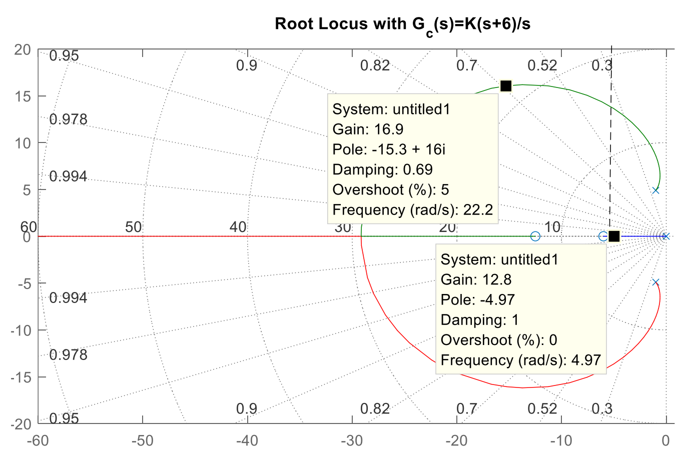
Q4: What closed loop specifications might we make?

(S1): Type 1 with ramp $e_{ss} \le 0.2$; (S2): $5\tau \le 1\,\text{sec.}$; (S3): max. overshoot <0.05.

Q5: Are the specifications achievable?



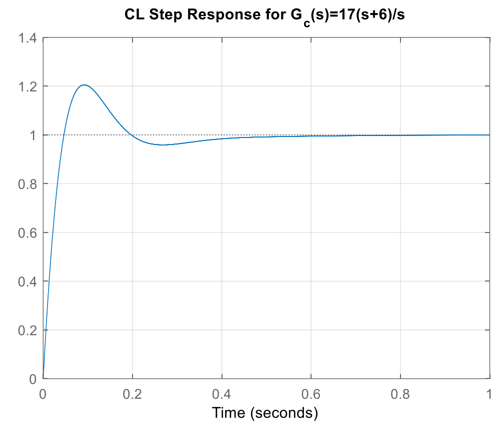
(S1): Type 1 with ramp $e_{ss} \le 0.2$: This will require a controller with a zero at the origin. It will also require that $\lim_{s\to 0} sG_c(s)G_p(s) = \lim_{s\to 0} sG_c(s) \ge 5$. A root locus plot of $(K/s)G_p(s)$ is shown at right. We see that pure integral control is not appropriate. The plant zero is so far away from the i-axis that it cannot attract the loci into the LHP. Hence, the controller will need to include a zero that can better attract the loci. However, it must be placed to the left of $s = -5 + i0$ so that the real root might achieve (S2).



Let's try $G_c(s) = \dfrac{K(s+6)}{s}$. The root locus at right shows that this controller might just work. With a controller gain of at least 16.9 (S3) is satisfied. For gains in this region the real CL pole will be to the left of the $\tau = 0.2$ line.

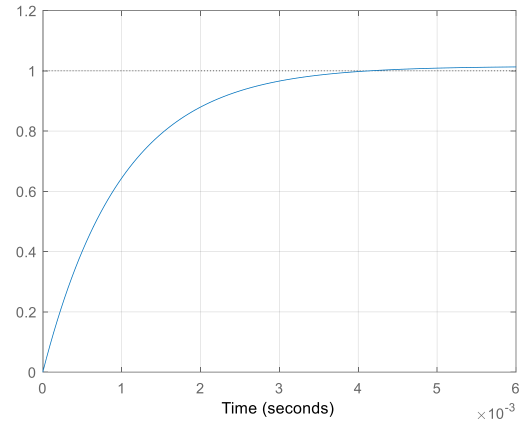
For $K = 17$, $\lim_{s\to 0} sG_c(s) = \lim_{s\to 0} s\left(\dfrac{K(s+6)}{6}\right) = 6K = 6(17) = 102$ will clearly satisfy $e_{ss} \le 0.2$.

Let's now see if the controller $G_c(s) = \frac{17(s+6)}{s}$ does the trick.



The plot at right shows that while (S2) is satisfied, (S3) is not. We could plot the ramp steady state error. However, we can compute it directly from $W(s) = \frac{34s^2 + 629s + 2550}{s^3 + 36s^2 + 654s + 2550}$: $\Delta(s) = 1 - W(s) = \frac{34s^2 + 25s}{s^3 + 36s^2 + 654s + 2550}$.
Hence: $e_{ss} \lim_{s \to 0} s\Delta(s)/s^2 = 25/2550 \cong 0.01$.

So, our problem lies with the excessive overshoot. Clearly, this is due to the closed loop zeros.



We could go into a black hole in investigating how to modify the controller so that the closed loop zeros would have a reduced effect. OR- we could try to simply crank up the controller gain and see what happens. The CL step response for K=30 is shown at right.

What we find is that not only do we have zero overshoot; we have also dramatically reduced the response time.

How did this happen?

The CL zeros and poles are:

z = -12.5 ; -6.0

p = -1003 ; -12.3 ; -6.0

What we see is that the closed loop transfer function as a pole/zero cancellation. And so, it behaves like the order system: $W_{r1}(s) \cong \frac{987(s+12.5)}{(s+1003)(s+12.3)}$. In fact, this reduced order system has an almost second pole/zero cancellation. Hence, its dynamics can be approximated by $W_{r2}(s) \cong \frac{1003}{s+1003}$.

Conclusion: Life can be 'complicated'. And sometimes it works out for the better. However, having said that, we did pay a price; namely, we needed a more powerful controller. One can try to minimize the controller power by trying to achieve the boundaries of the specifications. But if one has the power, then one can try to improve upon them. It should be clear from this example, that controller design is as much an art as it is a science. As an art form, some may find it beautiful, while others may find it repulsive. In either case, real art involves creativity. With the accelerating developments of the day, engineers would do well to strive for more creativity. However, it should be founded on science. It should not be mindless brainstorming.