Lecture 5 System Steady State Performance and Type Number

Consider a ***stable*** system with transfer function $W(s)$. Recall that, because $W(s)$ is stable, it follows that every *bounded input* will result in a *bounded output.* In particular, suppose the input is a step with amplitude U_o; that is, $u(t) = U_o 1(t)$. Then, in the *s*-domain, the output is $Y(s) = W(s)U(s) = W(s)\frac{U_o}{s}$. Since, here, we are interested in the *steady state* response, we know that if the system static gain is $W(0)$, then the steady state response will be $y_{ss}(t) = W(0)U_o$. This result demonstrates what is known as

The Final Value Theorem (FVT) Suppose that $\lim_{t\to\infty} y(t)$ exists. Call this final value y_{ss}. Let $Y(s)$ be the Laplace transform of $y(t)$. Then $y_{ss} = \lim_{s\to 0} sY(s)$.

Notice that, in the case of a step input, the 1/*s* in its Laplace transform is cancelled by the *s* in the final value theorem to give the result $y_{ss} = W(0)U_o$.

Now, let's assume that the system $W(s)$ is a *tracking system*. Specifically,

Definition 1. A *tracking system* is one wherein it is desired that the output track the input. Such a system is also referred to as a *command system* (i.e. the input is the command, and the output is the response to the command.), or a *reference system*. [**Notation:** In the case of a tracking system, it is customary to use the notation $y_c(t)$ (or, $y_R(t)$) to denote the input, and $y(t)$ to denote the output.]

In an ideal world a tracking (or command) system will exhibit perfect tracking (or, command response) in the steady state; that is, the *steady state error* $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} y_R(t) - y(t) \quad \equiv 0$. In words, the error is ***the difference between <u>what you want and what you get</u>.***

Before going further, let's return to the above system $W(s)$. Suppose that $W(s)$ is a stable tracking system. Then for a command input $y_R(t) = Y_o 1(t)$, the steady state output is $y_{ss} = W(0)Y_o$. Consequently, the steady state error is $e_{ss} = Y_o - W(0)Y_o = [1 - W(0)]Y_o$. And so, this system will be able to track a constant, Y_o, *perfectly* if and only if $W(0) = 1$; that is, the stable system $W(s)$ has unity static gain.

Now, suppose that the steady state tracking error for the step input is small, but not zero. Specifically, suppose that $e_{ss} = Y_o - W(0)Y_o = [1 - W(0)]Y_o = \varepsilon Y_o$ for some $\varepsilon \approx 0$. We now address the following:

Question: *How well can the system track a <u>ramp</u> input?*

Answer: To answer this question, let's define the *error transfer function*:

Definition 2. For a tracking system, the *error transfer function* is $\Delta(s) \overset{\Delta}{=} E(s)/Y_R(s)$, where $E(s)$ is the Laplace transform of the error $e(t) = y_R(t) - y(t)$, and $Y_R(s)$ is the Laplace transform of $y_R(t)$.

With this definition, in the *s*-domain we have:

$$\Delta(s) = \frac{E(s)}{Y_R(s)} = \frac{Y_R(s) - Y(s)}{Y_R(s)} = 1 - \frac{Y(s)}{Y_R(s)} = 1 - W(s) . \qquad (1)$$

Hence, for a ramp input $y_r(t) = V_o t$ with Laplace transform $Y_r(s) = V_o / s^2$, the error in the *s*-domain is

$$E(s) = \Delta(s)Y_R(s) = [1 - W(s)]\frac{V_o}{s^2}. \qquad (2)$$

Applying the *FVT* to (2), we obtain

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}[1 - W(s)]\frac{V_o}{s} = \lim_{s\to 0}\left[\frac{1 - W(s)}{s}\right]V_o = \lim_{s\to 0}\left[\frac{\Delta(s)}{s}\right]V_o = \lim_{s\to 0}\left[\frac{\varepsilon}{s}\right]V_o = \infty. \qquad (3)$$

We can draw a number of conclusions from (3):

(C1): Even though the steady state error to a *step* was finite, because it was not zero, the system is not able to track a *ramp* with finite steady state error.

(C2) Had the error transfer function $\Delta(s) = 1 - W(s)$ included a *zero* at the origin (i.e. an *s* term in the numerator polynomial), the system would have been able to track a ramp with finite steady state error. In this case, the tracking error for a unit step would have been *zero*.

We are now positioned to define what we mean by the *type number* of a system.

Definition 3. A *type-n* system is a system that exhibits finite, **non-zero** steady state error for an input that is an n^{th} degree polynomial in *t*.

In view of this definition, we can conclude the following important result:

Important Result #1: A *command tracking system* , $W(s)$, is a *type-n* system if and only if the error transfer function $\Delta(s) = 1 - W(s)$ has *n* zeros at the origin.

Now lets' get a bit more specific. Suppose that a tracking system $W(s)$ has the form

$$W(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots b_1 s + b_o}{a_n s^n + a_{m-1}s^{n-1} + \cdots a_1 s + a_o} \quad \text{(where } m \le n\text{)}.$$

Then the error transfer function is

$$\Delta(s) = 1 - W(s) = 1 - \frac{B(s)}{A(s)} = \frac{a_n s^n + \cdots a_{m+1}s^{m+1} + (a_m - b_m)s^m + (a_{m-1} - b_{m-1})s^{m-1} + \cdots (a_1 - b_1)s + (a_o - b_o)}{a_n s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_o}$$

Hence, $W(s)$ will be a *type-m* system if and only if only the coefficients of the lowest *m*-1 powers of *s* for $A(s)$ and $B(s)$ are identical. As a special case, if we only have $a_o = b_o$, then the system is *type*-1. [Think about it. If $a_o = b_o$, then $W(0) = 1$. Hence, the system can track a step perfectly.]

Now, let's assume that $W(s)$ IS a *type-m* system. Then

$$\Delta(s)=1-W(s)=1-\frac{B(s)}{A(s)}=\frac{a_n s^n+\cdots a_{m+1}s^{m+1}+(a_m-b_m)s^m}{a_n s^n+a_{n-1}s^{n-1}+\cdots a_1 s+a_o}.$$

Since we have assumed that $m\le n$, we can factor out the s^m term, so that:

$$\Delta(s)=\frac{[a_n s^{n-m}+\cdots a_{m+1}s+(a_m-b_m)]s^m}{a_n s^n+a_{n-1}s^{n-1}+\cdots a_1 s+a_o}.$$

Hence, we can clearly see that $\Delta(s)$ has *m* zeros at the origin.

If we give it an input of the form $y_R(t)=t^{m-1}$, then $Y(s)=(m-1)!/s^m$, and so

$$\lim_{t\to\infty}e(t)=\lim_{s\to 0}s[\Delta(s)Y_R(s)]=\lim_{s\to 0}s\left[\frac{a_n s^n+\cdots a_{m+1}s^{m+1}+(a_m-b_m)s^m}{a_n s^n+a_{n-1}s^{n-1}+\cdots a_1 s+a_o}\right]\left[\frac{(m-1)!}{s^m}\right]=0.$$

If we give it an input of the form $y_R(t)=t^m$, then $Y(s)=m!/s^{m+1}$, and so

$$\lim_{t\to\infty}e(t)=\lim_{s\to 0}s[\Delta(s)Y_R(s)]=\lim_{s\to 0}s\left[\frac{a_n s^n+\cdots a_{m+1}s^{m+1}+(a_m-b_m)s^m}{a_n s^n+a_{n-1}s^{n-1}+\cdots a_1 s+a_o}\right]\left[\frac{m!}{s^{m+1}}\right]=\frac{(a_m-b_m)m!}{a_o}.$$

Notice what took place. The Laplace transform of $y_R(t)=t^m$ that is $Y(s)=m!/s^{m+1}$ has a denominator term s^{m+1}. Multiplying this term by the s term in the FVT reduces it to s^m. This term cancels the s^m term in $\Delta(s)$. Pretty cool, eh? [I'm practicing my Minnesota accent. ☺] From the above, we have a second important result.

Important Result #2: A *type-m* *command tracking system*, $W(s)$, will have **zero** steady state error
For any input $y_R(t)$ that is a polynomial of degree **less than** *m*. Furthermore, the steady state error
for a polynomial of degree *m* will be proportional to the ratio $(a_m-b_m)/a_o$.

Example 1. (4.11 on p.210) Consider a plant with $G_p(s)=\dfrac{1}{s^2+2\xi s+1}$, along with a forward loop controller of the form $G_c(s)=\dfrac{K(s+\alpha)}{s+\beta}$. We would like to see how well this controller can work in a *unity feedback* closed loop *tracking system.*

(a) Find the constraints on the controller parameters such that the closed loop system will be stable.

Solution: The closed loop characteristic polynomial is: $D(s)=(s^2+2\xi s+1)(s+\beta)+K(s+\alpha)$, or

$$D(s)=s^3+(2\xi+\beta)s^2+(2\xi\beta+K+1)s+(K\alpha+\beta).$$

The Routh array for $D(s)$ is:

$$
\begin{array}{llll}
s^3: & 1 & 2\xi\beta + K + 1 & \\
s^2: & 2\xi + \beta & K\alpha + \beta & \text{where } c = \dfrac{(2\xi+\beta)(2\xi\beta+K+1)-(K\alpha+\beta)}{(2\xi+\beta)} \\
s^1: & c & 0 & \\
s^0: & K\alpha + \beta & 0 &
\end{array}
$$

If we assume that all of the controller parameters are greater than zero, then there will be no sign changes in the first column of the array if and only if $c > 0$. This condition could be simplified a bit, but it will still be rather complicated. Hence, we will not pursue it any further here.

(b) Assuming the system is stable, find the constraints on K, α and β such that the closed loop system will be *Type*-1.

Solution: The closed loop transfer function is $W(s) = \dfrac{N(s)}{D(s)} = \dfrac{Ks + K\alpha}{s^3 + (2\xi+\beta)s^2 + (2\xi\beta + K + 1)s + (K\alpha + \beta)}$

This system will be *Type*-1 if the following condition holds: (C1): $K\alpha = K\alpha + \beta$

The condition (C1) requires that $\beta = 0$. In this case the closed loop system will have unity static gain. Hence, it will be at least *Type*-1. To make it *Type*-2 would require that the following condition also holds: $K = 2\xi\beta + K + 1 = 2\xi(0) + K + 1 = K + 1$. Since this cannot hold, the best that can be achieved as a *Type*-1 system for $\beta = 0$. In this case, we have the closed loop system

$$W(s) = \frac{N(s)}{D(s)} = \frac{Ks + K\alpha}{s^3 + 2\xi s^2 + (K+1)s + K\alpha}$$

with error transfer function

$$\Delta(s) = 1 - W(s) = \frac{s(s^2 + 2\xi s + 1)}{s^3 + 2\xi s^2 + (K+1)s + K\alpha}.$$

And so, for a command input of the form $y_R(t) = c_0\, t$, and Laplace transform $Y_R(s) = c_0 / s^2$, the steady state tracking error becomes

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\,\Delta(s)\left(\frac{c_0}{s^2}\right) = \lim_{s\to 0} s\left[\frac{s(s^2 + 2\xi s + 1)}{s^3 + 2\xi s^2 + (K+1)s + K\alpha}\right]\left(\frac{c_0}{s^2}\right) = \frac{c_0}{K\alpha}.$$

For $\beta = 0$ the controller is: $G_c(s) = \dfrac{K(s+\alpha)}{s} = K + \dfrac{K\alpha}{s} = K_p + \dfrac{K_i}{s}$

Hence, the quantity $K\alpha$ is simply the gain of the *integral* portion of the P-I controller. The larger this gain is, the smaller the steady state tracking error will be.

(c) Assuming the condition for a *Type*-1 system holds. Then from (a) we have:

$$W(s) = \frac{N(s)}{D(s)} = \frac{Ks + K\alpha}{s^3 + 2\xi s^2 + (K+1)s + K\alpha}$$

Find more specific conditions on the controller parameters to ensure stability.

Solution: For $\beta = 0$ the Routh array condition for closed loop stability becomes:

$$c = \frac{(2\xi + \beta)(2\xi\beta + K + 1) - (K\alpha + \beta)}{(2\xi + \beta)} = \frac{(2\xi)(K+1) - K\alpha}{2\xi} > 0 .$$

This, in turn, requires that $2\xi(K+1) - K\alpha > 0 \quad \Leftrightarrow (2\xi - \alpha)K > -2\xi \quad \Leftrightarrow \quad (\alpha - 2\xi)K < 2\xi$.
If we assume that $\alpha > 2\xi$, then we have the following conditions for a *Type*-1 closed loop system:

The controller must (i) have the form $G_c(s) = \frac{K(s+\alpha)}{s} = K + \frac{K\alpha}{s}$ (i.e. it must be a P-I controller), (ii) $K < 2\xi/(\alpha - 2\xi)$, and (iii) $\alpha > 2\xi$

Remark: If we express the controller parameters as $G_c(s) = K + \frac{K\alpha}{s} = K_p + \frac{K_i}{s}$, then the condition for stability becomes: $2\xi(K_p + 1) - K_i > 0$. While this condition highlights the relation between the proportional and integral gains needed for stability, it turns out that the form of the controller that highlights its *zero* is much more useful in design (especially when we cover the *Root Locus* method of control system design). □

We will now arrive at a classic defining characteristic of a *Type-n* *unity feedback command system*.

Such a system will have the transfer function (assuming that $G(s)$ is a ratio of polynomials):

$$W(s) = \frac{G(s)}{1+G(s)} = \frac{z(s)/p(s)}{1+z(s)/p(s)} = \frac{z(s)}{p(s)+z(s)} .$$

Hence, the error transfer function is:

$$\Delta(s) = 1 - W(s) = 1 - \frac{G(s)}{1+G(s)} = \frac{1}{1+G(s)} = \frac{1}{1+z(s)/p(s)} = \frac{p(s)}{p(s)+z(s)} .$$

Because this is a *Type-n* command system, the error transfer function, $\Delta(s)$ has exactly *n* zeroes at the origin. But the zeroes of this transfer function are exactly the poles of the open loop $G(s)$.

Suppose this system is a *Type-n* system. Then for an input of the form $y_R(t) = t^n$, with $Y(s) = n!/s^{n+1}$, the steady state error is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s[\Delta(s)Y_R(s)] = \lim_{s\to 0} s\left[\frac{1}{1+G(s)}\right]\left[\frac{n!}{s^{n+1}}\right] = \frac{n!}{\lim_{s\to 0} s^n G(s)} \stackrel{\Delta}{=} \frac{n!}{K_{error}^{(n)}}$$

where we have defined the 'error constant $K_{error}^{(n)} \overset{\Delta}{=} \lim_{s \to 0} s^n G(s)$. And so, we arrive at a third important result.

Important Result #3: The *type* number of a stable <u>unity feedback command *tracking system*</u> , $W(s)$, is exactly equal to the number of **open loop poles** at the origin. Furthermore, the steady state error for the input $y_R(t) = t^n$ to a *Type-n* system of this form is simply $e_{ss} = n!/\lim_{s \to 0} s^n G(s)$.

The Type Number for a Disturbance System

When the input to a system is not a command, but rather an unwanted disturbance, the system is said to be a disturbance system. Recall that our definition of 'error' is: ***the difference between <u>what you want and what you get</u>.***

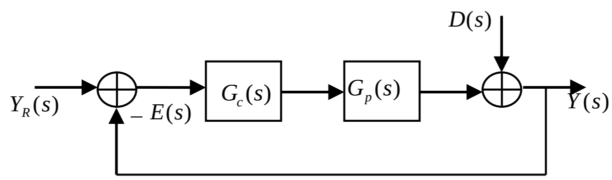
Question: What is the error transfer function associated with a disturbance system $W(s)$?

Answer: What we want is that the disturbance will not be seen at all in the output. Hence, the error transfer function is $\Delta(s) = 0 - W(s) = -W(s)$.

With a little thought it should become clear that the error transfer function given in the answer above must entail all the same properties that we developed for a command system error transfer function. In particular, **the disturbance system $W(s)$ will be a *type-n* disturbance system if it has *n* zeroes at the origin.**

Example 1 (continued): Suppose that there is a disturbance input located at the output of the plant, and that the closed loop is a *type*-1 tracking system. Determine its type number as a disturbance system.

Solution: The system block diagram is shown at the right. The disturbance transfer function is



$$W_d(s) \overset{\Delta}{=} \frac{1}{1+G_c(s)G_p(s)} = \frac{1}{1+\left[\frac{K(s+\alpha)}{s}\right]\left[\frac{1}{s^2+2\xi s+1}\right]} = \frac{s(s^2+2\xi s+1)}{s(s^2+2\xi s+1)+K(s+\alpha)}.$$

Hence, the disturbance error transfer function is exactly the same as the command system error transfer function; namely:

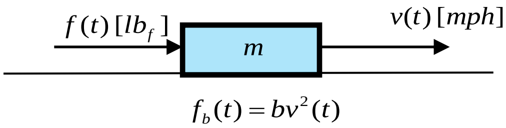
$$\Delta(s) = -\frac{s(s^2+2\xi s+1)}{s(s^2+2\xi s+1)+K(s+\alpha)}.$$

Because the error transfer function has one zero at the origin, the disturbance system is also a *Type*-1 system. Furthermore, the steady state response to a disturbance input $d(t) = d_0 t$ is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\,\Delta(s)\left(\frac{d_0}{s^2}\right) = \lim_{s\to 0} \cancel{s}\left[\frac{-\cancel{s}(s^2+2\xi s+1)}{s^3+2\xi s^2+(K+1)s+K\alpha}\right]\left(\frac{d_0}{\cancel{s}^2}\right) = \frac{-d_0}{K\alpha}\,.$$

We will not pursue the properties of a unity-feedback disturbance system, since commonly the disturbance transfer function is not of the unity-feedback type. □

***Example* 2. A simple model for automobile cruise control.**



$$\sum_{forces} = m\dot{v} = f - bv^2$$

The force balance equation gives: $m\dot{v}(t) + bv^2(t) = f(t) \quad ; \quad v(0_-) = v_o$. (1)

***Linearization of* (1):** If we let $g(v) \overset{\Delta}{=} bv^2$, then the slope of $g(v)$ at the initial condition speed is $g(v_o) = 2bv_o$. Hence, for small speed variations about v_o we can approximate $g(v)$ as:

$$g(v) \cong g(v_o) + g'(v_o)(v - v_o) \quad = \quad bv_o^2 + 2bv_o(v - v_o)\,. \qquad (2)$$

Substituting (2) into (1) gives: $m\dot{v}(t) + bv_o^2 + 2bv_o[v(t) - v_o] = f(t) \quad ; \quad v(0_-) = v_o$. (3)

Define the velocity relative to v_o as $\upsilon(t) \overset{\Delta}{=} v(t) - v_o$. Then $\dot{\upsilon}(t) = \dot{v}(t)$. Furthermore, we can also define the force relative to that required to maintain the velocity v_o as $\delta f(t) \overset{\Delta}{=} f(t) - bv_o^2$. We then have the *relative force/velocity* model:

$$m\dot{\upsilon}(t) + \beta\,\upsilon(t) = \delta f(t) \quad ; \quad \upsilon(0_-) = 0\,, \qquad (4a)$$

where $\beta = \beta(v_o) \overset{\Delta}{=} 2bv_o$. (4b)

Remark 1. Notice that (4) is the model that describes the *relative force/speed relationship* about the speed v_o , and that the parameter $\beta = \beta(v_o) \overset{\Delta}{=} 2bv_o$ depends on that speed. Even though (4) is a constant-coefficient linear differential equation, one needs to be aware that the parameter β depends upon the nominal speed at which the vehicle is traveling. It is, perhaps, for this reason that the authors ignored the initial condition in their model. It is my personal opinion that they should not have done so. For, as (4) reveals, any *cruise control* system that is implemented will be required to modify the controller parameters so that they correspond to the possibly changing value of v_o .■

Taking the Laplace transform of (4) gives: $msV(s) + \beta V(s) \quad = \quad \delta F(s)$. Hence,

$$V(s) \quad = \quad G_p(s)\,\delta F(s)\,. \qquad (5a)$$

where the plant transfer function is: $G_p(s) \overset{\Delta}{=} \dfrac{1}{ms+\beta}$ (5b)

The block diagram below includes both a command speed input, as well as a force disturbance input. The command system is unity-feedback, while the disturbance system is not.

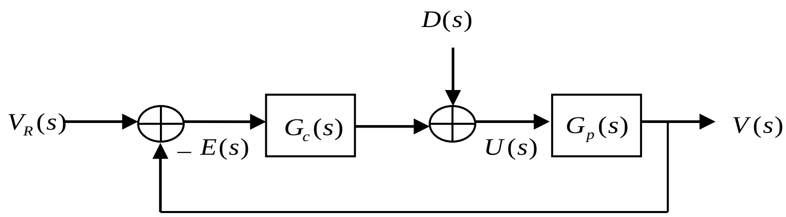


Figure 1. Block diagram of a cruise control feedback control system.

Problem: If it is possible, design a controller that will achieve (i) a *Type*-1 command system, (ii) a *Type*-1 disturbance system, and (iii) controller gain(s) that will permit both system steady state errors to be achieved at given levels. Also, for specified CL parameters τ and ζ arrive at solutions for controller parameters.

Solution: Clearly, integral must be included in $G_c(s)$, as it will result in an open loop pole at the origin. Using $G_c(s) = \dfrac{K_d s^2 + K_p s + K_i}{s}$ gives $G_c(s) = G_c(s)G_p(s) = \dfrac{K_d s^2 + K_p s + K_i}{s(ms+\beta)}$. Hence, for the unity feedback command system, the error constant is $K_{error}^{(1)} \overset{\Delta}{=} \lim_{s\to 0} s^1 G(s) = K_i / \beta$. For a unit ramp input the command system *ss* error is then $e_{ss}^c = \beta / K_i$. The disturbance CL transfer function is $W_d(s) = \dfrac{G_p(s)}{1+G_c(s)G_p(s)} = \dfrac{s}{(K_d+m)s^2 + (K_p+\beta)s + K_i}$. The associated error transfer function is $\Delta_d(s) = \dfrac{-s}{(K_d+m)s^2+(K_p+\beta)s+K_i}$. And so, this is also Type-1. For a disturbance input $d(t) = t$ with Laplace transform $D(s) = 1/s^2$ the ss error is $e_{ss}^d = \lim_{s\to 0} s\Delta(s)D(s) = 1/K_i$.

Now suppose that we require that $e_{ss}^c = 1.0\ mph$. This requires $K_i = \beta$. The command CL transfer function is then:

$$W_c(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} = \frac{K_d s^2 + K_p s + \beta}{(K_d+m)s^2 + (K_p+\beta)s + \beta}.$$

The CL characteristic polynomial is: $p(s) = s^2 + \left(\dfrac{K_p+\beta}{K_d+m}\right)s + \left(\dfrac{\beta}{K_d+m}\right)$. Now we will specify τ and ζ.

Since $\tau = 1/\zeta\omega_n$, these specifications result in a specification for $\omega_n = 1/\tau\zeta$. Since $\omega_n^2 = \dfrac{\beta}{K_d+m}$, we can find the needed value for K_d. Finally, since $2\zeta\omega_n = \dfrac{K_p+\beta}{K_d+m}$, we can find the needed value for K_p. □