LECTURE 4 Block Diagrams, Poles/Zeros, and PID Control for Tracking & Disturbance Rejection

In *Example* 1 of LECTURE 3 we addressed the feedback control system at right. In relation to this, it was stated:

Clearly, $W(s) = \dfrac{Y(s)}{Y_R(s)} = \dfrac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)}$. (1)

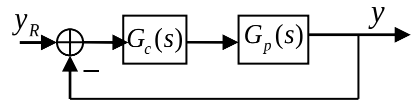


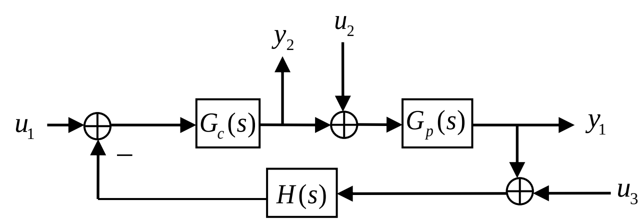
Figure 1 Unity feedback block diagram.

The derivation of (1) was done at the whiteboard: The input to $G_c(s)$ is $Y_R(s)-Y(s)$. Hence, the output from $G_c(s)$ is $G_c(s)[Y_R(s)-Y(s)]$. But this, in turn, is the input to $G_p(s)$. And the output of $G_p(s)$ is $Y(s)$. Hence,

$Y(s) = G_p(s)G_c(s)[Y_R(s)-Y(s)]$. Gathering terms gives: $[1+G_p(s)G_c(s)]Y(s) = G_p(s)G_c(s)Y_R(s)$, which gives (1).

We will now consider the more general block diagram at right:

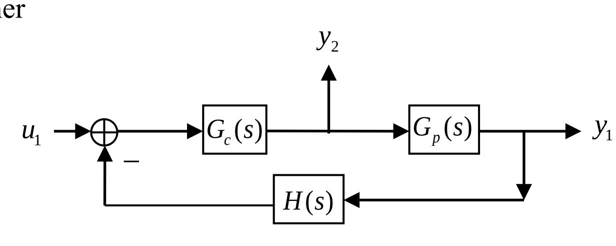
This is a 3-input/2-output system.



(i) To find transfer functions related to input u_1, we 'turn off' all other inputs as shown at right.

$$Y_2 = G_c(U_1 - G_pHY_2) \Rightarrow \frac{Y_2}{U_1} = \frac{G_c}{1+G_pH}.$$

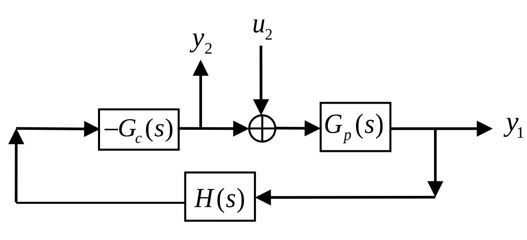
$$Y_1 = G_cG_p(U_1 - HY_1) \Rightarrow \frac{Y_1}{U_1} = \frac{G_cG_p}{1+G_cG_pH}$$



(ii) To find transfer functions related to input u_2, we 'turn off' all other inputs off, as shown at right.

$$Y_2 = -G_cHG_p(U_2 + Y_2) \Rightarrow \frac{Y_2}{U_2} = \frac{-G_cG_pH}{1+G_cG_pH}$$

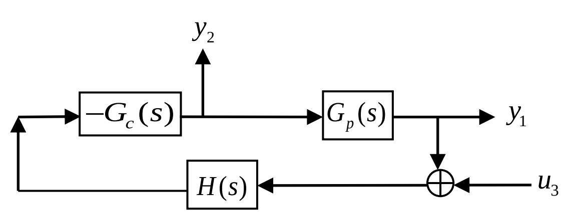
$$Y_1 = G_p(U_2 - G_cHY_2) \Rightarrow \frac{Y_1}{U_2} = \frac{G_p}{1+G_cG_pH}$$



(iii) To find transfer functions related to input u_3, we 'turn off' all other inputs off, as shown at right.

$$Y_2 = -G_cH(U_3 + G_pY_2) \Rightarrow \frac{Y_2}{U_3} = \frac{-G_cH}{1+G_cG_pH}$$

$$Y_1 = -G_cG_pH(U_2 + Y_2) \Rightarrow \frac{Y_1}{U_2} = \frac{-G_cG_pH}{1+G_cG_pH}$$

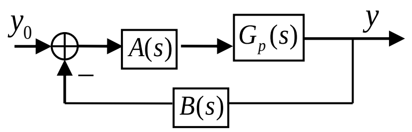


Notice that all transfer functions have the same denominator $1+G_c(s)G_p(s)H(s)$. Consequently, they all have the same poles. If all the poles are in the LHP, then all are stable systems. On the other hand they have different numerators. Consequently, they do not all have the same zeros. While the zeros can have a notable effect on the responses, those responses are mainly controlled by the system poles.

Tracking versus Disturbance Rejection

A tracking system is one where the input is a command input and the output is desired to track it. A disturbance rejection system is one where the input is a disturbance and the output is desired to be oblivious to it.

Consider the feedback control system block diagram shown at right.
The closed loop transfer function is:



$$\frac{Y(s)}{Y_0(s)} = W(s) = \frac{A(s)G_p(s)}{1 + A(s)G_p(s)B(s)}.$$

For a tracking system we want $A(s) = G_c(s)$ and $B(s) = 1$. For obvious reasons this is called a *unity feedback* control system. We want $y(t)$ to track the command input $y_c(t)$.

For a disturbance rejection system we want $A(s) = 1$ and $B(s) = G_c(s)$. We want $y(t)$ to be oblivious to the disturbance input $y_d(t)$.

Example 1 Suppose that $G_p(s) = \dfrac{1}{s+0.1}$. This is a first order plant with $\tau = 10$ sec and $g_s = 10$. Consider the simple proportional controller $G_c(s) = K$.

For command input $y_c(t)$, placing the controller in the forward loop gives: $\dfrac{Y(s)}{Y_c(s)} = W_c(s) = \dfrac{K}{s + (0.1+K)}$. The closed loop time constant and static gain are $\tau = 1/(0.1+K)$ and $g_s = K/(0.1+K)$.

For disturbance input $y_d(t)$, placing the controller in the feedback loop gives: $\dfrac{Y(s)}{Y_d(s)} = W_d(s) = \dfrac{1}{s + (0.1+K)}$. The closed loop time constant and static gain are $\tau = 1/(0.1+K)$ and $g_s = 1/(0.1+K)$.

Both systems have the same poles. However, their static gains are markedly different. For a unit step input, the steady state output of the tracking system is $g_s = K/(0.1+K)$; whereas the steady state output of the disturbance rejection system is $g_s = 1/(0.1+K)$.

Suppose that we choose $K = 100$. Then the tracking system steady state output is $g_s = 100/100.1 \cong 1$. We have nearly perfect tracking. The steady state output for the disturbance system is $g_s = 1/100.1 \cong 0.01$. Hence, the steady state output is relatively oblivious to the disturbance input. □

It is not at all inconceivable that a single control can accomplish both improved tracking response and greater disturbance rejection. This feat is illustrated in the next example.

Example 2 It is desired to design a feedback control system that can improve the dynamic response of the short period longitudinal dynamics of a certain aircraft, while at the same time making the response more robust to wind gusts. Suppose that the uncontrolled dynamics are modeled via the transfer function:

$$\frac{\theta(s)}{F(s)} = G_p(s) = \frac{50}{s^2 + s + 25}. \tag{1}$$

where $\theta(t)$ is the pitching response to a vertical force $f(t)$. The unit step response is shown at right. Clearly, it is unacceptable.

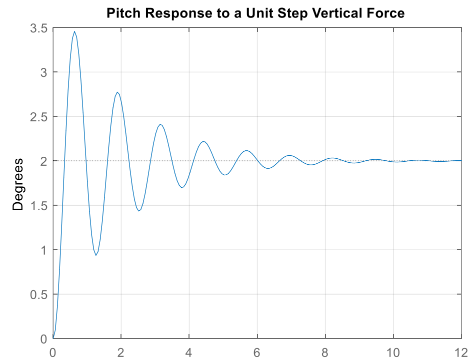
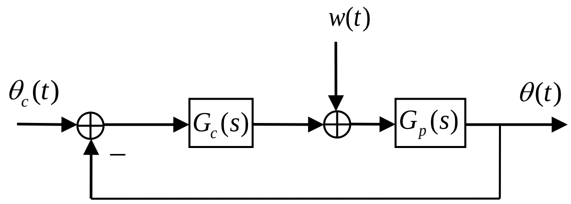


Figure 1 Pitch response to a unit step command.

The closed loop feedback control system to be implemented is shown at right. The command input is $\theta_c(t)$, and the disturbance input is the wind force $w(t)$. The command and disturbance transfer functions are:



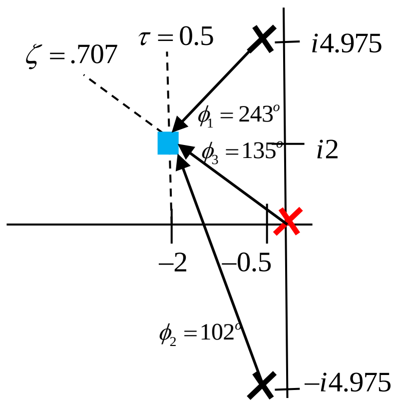
$$\frac{\theta(s)}{\delta_e(s)} = W_c(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} \text{ and } \frac{\theta(s)}{w(s)} = W_d(s) = \frac{G_p(s)}{1+G_c(s)G_p(s)},$$

respectively. We see that the command system is unity feedback, while the disturbance rejection system includes the controller in the feedback loop.

We will now proceed to design a PID controller that achieves the following specifications:

(S1): Unity static gain. (S2) Optimal damping (i.e. $\zeta = 0.707$). (S3) 4τ response time equal to 2 seconds.

We will use the root locus pole-placement method to design the controller, in order to motivate future discussion of this method. To begin, we first determine the desired closed loop poles associated with (S2) and (S3). These specifications require that they lie at the intersection of the $\zeta = 0.707$ and $\tau = 0.5$ lines, as shown at right.



The black crosses are the plant poles, and the red cross is the controller pole associated with integral control needed to satisfy (S1). There will be two controller zeros. The ***root locus angle criterion*** states that the blue square will be a closed loop pole if: $(\theta_1+\theta_2)-(243^o+135^o+102^o) = -180^o$. From this, we find that the angles from the controller zeros to the closed loop pole must satisfy $\theta_1+\theta_2 = 300^o$.

With a little thought, for this condition to be satisfied, the zeros must be a complex-conjugate pair. Let's try controller zeros $z_{1,2} = 0.3 \pm i2$. Then the angle from $z_1 = 0.3+i2$ to the blue square is 180^o, and the angle from $z_2 = 0.3-i2$ to the blue square is 120^o. The total angle is 300^o. The controller then has the form

$$G_c(s) = \frac{K(s-z_1)(s-\bar{z}_1)}{s} = \frac{K[s^2-2\,\mathrm{Re}(z_1)s+|z_1|^2]}{s} = \frac{K(s^2-0.6s+4)}{s}. \tag{2}$$

The resulting open loop transfer function is $G_c(s)G_p(s)$. The associated closed loop root locus is shown at right. From the data cursor we see that for $K = 0.128$ the closed loop system has complex-conjugate poles at $-2 \pm i1.88$ with associated $\zeta = 0.73$ and $\tau = 0.5$. The third real pole at -3.4 has an associated $\tau = 0.3$, which is faster than $\tau = 0.5$. Hence, the controller is:

$$G_c(s) = \frac{0.128(s^2-0.6s+4)}{s}. \tag{3}$$

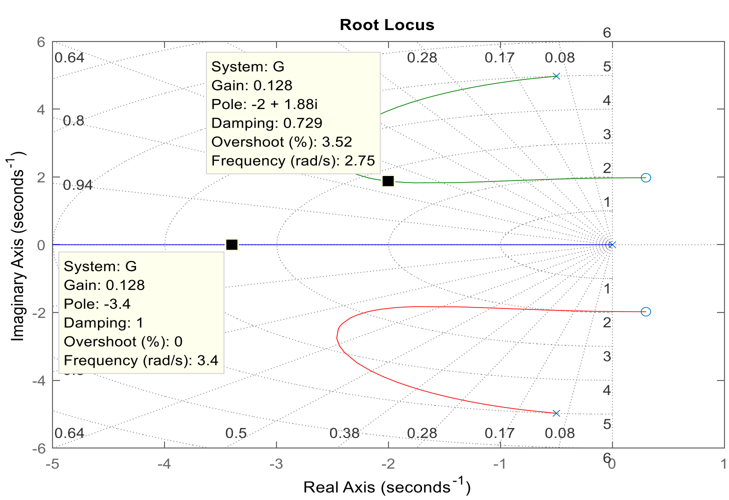


Figure 2 Closed loop root locus.

The closed loop transfer function is:

$$W_c(s) = \frac{6.4s^2 - 3.84s + 25.6}{s^3 + 7.4s^2 + 21.16s + 25.6}. \qquad (4)$$

Note that its zeros are those of the controller. To appreciate the influence of the closed loop zeros, let

$$W_c^*(s) = \frac{25.6}{s^3 + 7.4s^2 + 21.16s + 25.6}. \qquad (5)$$

The unit step responses of $G_p(s)$, $W_c(s)$, and $W_c^*(s)$ are shown at right.

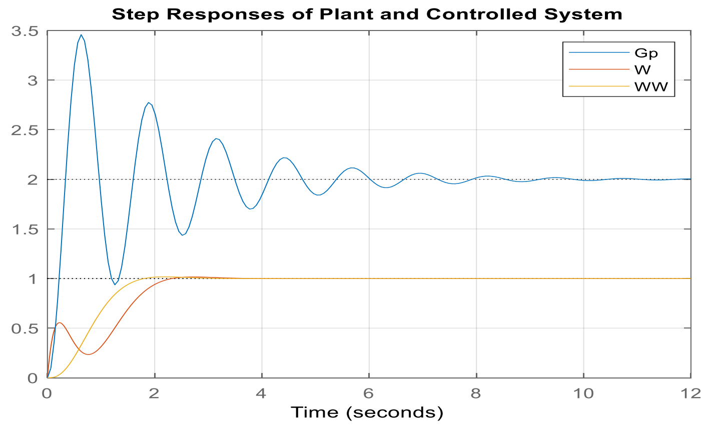


Figure 3 Unit step responses for $G_p(s)$, $W_c(s)$, and $W_c^*(s)$.

Clearly $W_c(s)$ is a significant improvement over $G_p(s)$. Indeed, all three specifications are satisfied. However, in view of the initial small oscillations, $W_c(s)$ is not as desirable as $W_c^*(s)$. The above specifications related only to the closed loop poles. The presence of the closed loop zeros was not taken into account in those specifications. Moreover, it is not easy to take the same into account.

In relation to the wind gust disturbance input we have the closed loop transfer function

$$W_d(s) = \frac{50s}{s^3 + 7.4s^2 + 21.16s + 25.6}. \qquad (6)$$

The response to a unit step gust is shown at right. The static gain is zero, and so the response is oblivious to the gust after ~4 seconds. However, the initial response to the gust is notable.

It should also be evident that the zero $s = 0$ is the reason that the static gain is $g_s = W_d(0) = 0$. To better appreciate the influence of the magnitude 50 in (6), consider

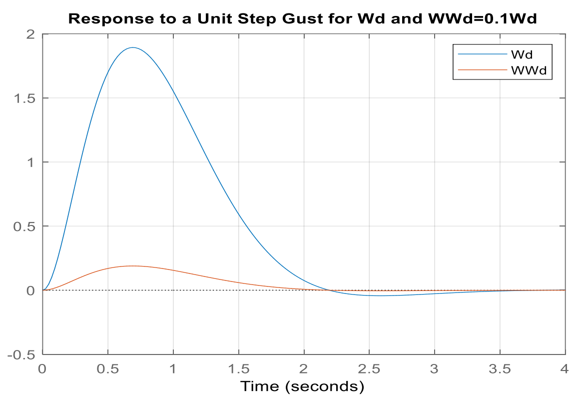


Figure 4 Unit step response for $W_d(s)$ and $W_d^*(s)$.

$$W_d^*(s) = \frac{5s}{s^3 + 7.4s^2 + 21.16s + 25.6}. \qquad (7)$$

The unit step response for $W_d^*(s)$ is also shown in Figure 4. Its peak is 1/10th that of $W_d(s)$. The differential equation associated with $W_d(s)$ is

$$\dddot{\theta} + 7.4\ddot{\theta} + 21.16\dot{\theta} + 25.6\theta = 50\dot{w}. \qquad (8)$$

In words, $\theta(t)$ is not responding to the unit step, itself. Rather, it is responding to its derivative; which is a Dirac delta function having an intensity equal to 50. This is a big impulse! It is, in part, for this reason that the authors devote the entire section 3.5 (pp.142-152) on the effects of zeros and additional poles. A term s^m in the numerator of a transfer function corresponds to the m^{th} derivative of the input. To further highlight such an influence, suppose that the input to (8) were a small amplitude sinusoid $\varepsilon\sin(\omega t)$. The derivative is $\varepsilon\omega\cos(\omega t)$. For a high frequency ω the amplitude could be yuge! (Thanks Bernie ☺).

Conclusion Aircraft transfer functions very often include zeros. Even if the controller contains no zeros, one must take care in requiring specifications that are based only on poles. Even if such specifications are achieved, it may well be that the actual dynamics are significantly different that those associated with the specified poles. □

Matlab Code for Example 2

```
%PROGRAM NAME: lec4EX2.m
Gc=tf([1 -.6 4],[1 0]);
G=Gc*Gp;
figure(101)
rlocus(G)
grid
%================
K=0.128;
Gc=K*Gc;
Wc=feedback(Gc*Gp,1);
figure(102)
step(Gp)
hold on
step(Wc)
title('Step Responses of Plant and Controlled System')
grid
%-------------
%Add feedback system without closed loop zeros:
[n,d]=tfdata(Wc,'v');
WWc=tf(n(4),d);
step(WWc)
legend('Gp','W','WW')
%=============
%Disturbance TFs:
Wd=feedback(Gp,Gc);
[n,d]=tfdata(Wd,'v');
figure(103)
step(Wd)
title('Response to a Unit Step Gust')
grid
WWd=0.1*Wd;
hold on
step(WWd)
title('Response to a Unit Step Gust for Wd and WWd=0.1Wd')
legend('Wd','WWd')
```