LECTURE 4 Example 2

The purpose of this LECTURE 4 addendum is to demonstrate how a little curiosity, along with a little trigonometry, can lead to fairly amazing results. Before we pursue those results, we repeat Example 2 as presented in LECTURE 4.

Example 2 It is desired to design a feedback control system that can improve the dynamic response of the short period longitudinal dynamics of a certain aircraft, while at the same time making the response more robust to wind gusts. Suppose that the uncontrolled dynamics are modeled via the transfer function:

$$\frac{\theta(s)}{F(s)} = G_p(s) = \frac{50}{s^2+s+25}. \qquad (1)$$

where $\theta(t)$ is the pitching response to a vertical force $f(t)$. The unit step response is shown at right. Clearly, it is unacceptable.

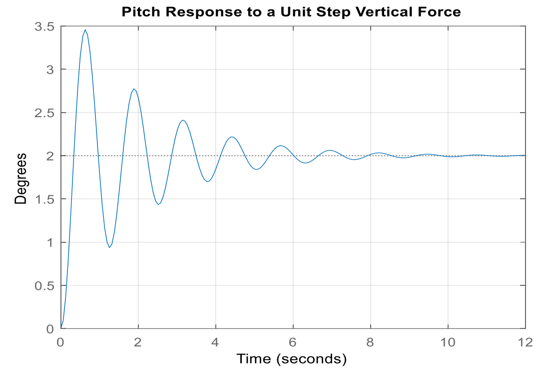


Figure 1 Pitch response to a unit step command.

The closed loop feedback control system to be implemented is shown at right. The command input is $\theta_c(t)$, and the disturbance input is the wind force $w(t)$. The command and disturbance transfer functions are:

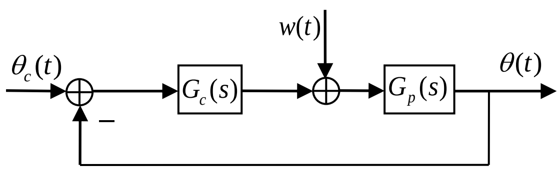


Figure 2 Feedback control block diagram.

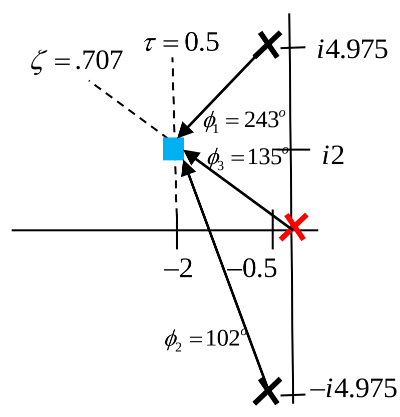
$$\frac{\theta(s)}{\delta_e(s)} = W_c(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} \text{ and } \frac{\theta(s)}{w(s)} = W_d(s) = \frac{G_p(s)}{1+G_c(s)G_p(s)},$$

respectively. We see that the command system is unity feedback, while the disturbance rejection system includes the controller in the feedback loop.

The closed loop specifications are:

(S1): Unity static gain. **(S2)** Optimal damping (i.e. $\zeta = 0.707$). **(S3)** 4τ response time no greater than 2 seconds.

We will use the root locus pole-placement method to design the controller, in order to motivate future discussion of this method. To begin, we first determine the desired closed loop poles associated with (S2) and (S3). These specifications require that they lie at the intersection of the $\zeta = 0.707$ and $\tau = 0.5$ lines, as shown at right.



The black crosses are the plant poles, and the red cross is the controller pole associated with integral control needed to satisfy (S1). There will be two controller zeros. The ***root locus angle criterion*** states that the blue square will be a closed loop pole if: $(\theta_1+\theta_2)-(243^o+135^o+102^o) = -180^o$. From this, we find that the angles from the controller zeros to the closed loop pole must satisfy $\theta_1+\theta_2 = 300^o$.

With a little thought, for this condition to be satisfied, the zeros must be a complex-conjugate pair. Let's try controller zeros $z_{1,2} = 0.3 \pm i2$. Then the angle from $z_1 = 0.3+i2$ to the blue square is 180^o, and the angle from $z_2 = 0.3-i2$ to the blue square is 120^o. The total angle is 300^o. The controller then has the form

$$G_c(s) = \frac{K(s-z_1)(s-\bar{z}_1)}{s} = \frac{K[s^2-2\,\mathrm{Re}(z_1)s+|z_1|^2]}{s} = \frac{K(s^2-0.6s+4)}{s}. \qquad (2)$$

The resulting open loop transfer function is $G_c(s)G_p(s)$. The associated closed loop root locus is shown at right. From the data cursor we see that for $K = 0.128$ the closed loop system has complex-conjugate poles at $-2 \pm i1.88$ with associated $\zeta = 0.73$ and $\tau = 0.5$. The third real pole at -3.4 has an associated $\tau = 0.3$, which is faster than $\tau = 0.5$. Hence, the controller is:

$$G_c(s) = \frac{0.128(s^2 - 0.6s + 4)}{s}. \qquad (3)$$

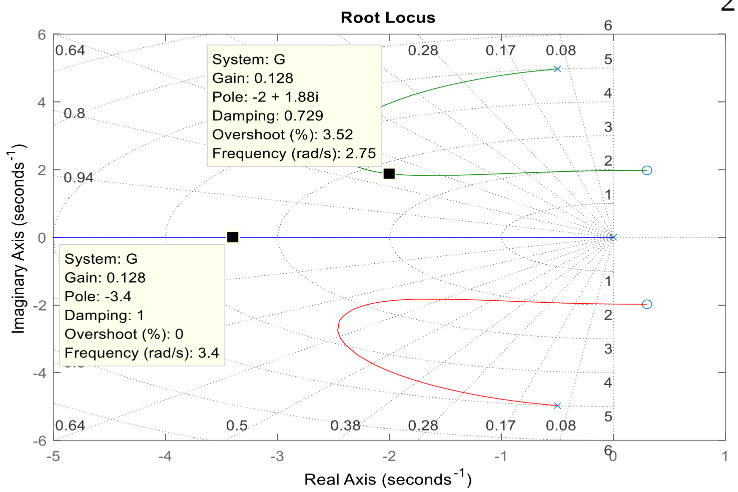


Figure 2 Closed loop root locus.

The closed loop transfer function is:

$$W_c(s) = \frac{6.4s^2 - 3.84s + 25.6}{s^3 + 7.4s^2 + 21.16s + 25.6}. \qquad (4)$$

Note that its zeros are those of the controller. To appreciate the influence of the closed loop zeros, let

$$W_c^*(s) = \frac{25.6}{s^3 + 7.4s^2 + 21.16s + 25.6}. \qquad (5)$$

The unit step responses of $G_p(s)$, $W_c(s)$, and $W_c^*(s)$ are shown at right.

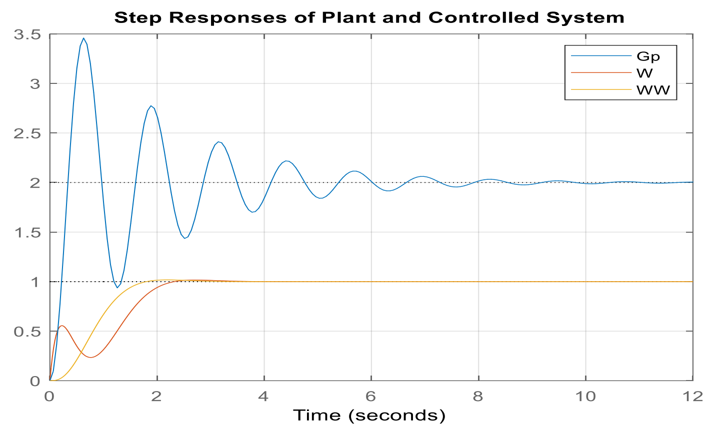


Figure 3 Unit step responses for $G_p(s)$, $W_c(s)$, and $W_c^*(s)$.

Clearly $W_c(s)$ is a significant improvement over $G_p(s)$. Indeed, all three specifications are satisfied. However, in view of the initial small oscillations, $W_c(s)$ is not as desirable as $W_c^*(s)$. The above specifications related only to the closed loop poles. The presence of the closed loop zeros was not taken into account in those specifications. Moreover, it is not easy to take the same into account.

In relation to the wind gust disturbance input we have the closed loop transfer function

$$W_d(s) = \frac{50s}{s^3 + 7.4s^2 + 21.16s + 25.6}. \qquad (6)$$

The response to a unit step gust is shown at right. The static gain is zero, and so the response is oblivious to the gust after ~4 seconds. However, the initial response to the gust is notable.

It should also be evident that the zero $s = 0$ is the reason that the static gain is $g_s = W_d(0) = 0$. To better appreciate the influence of the magnitude 50 in (6), consider

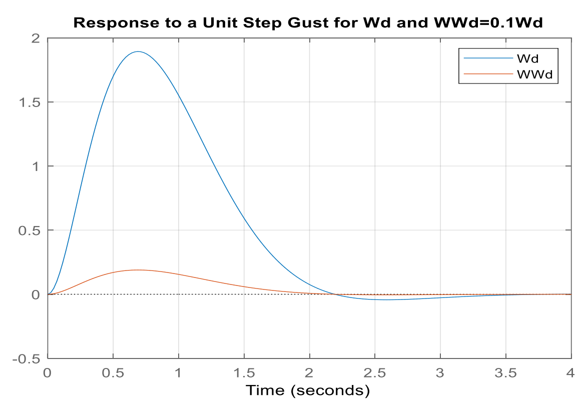


Figure 4 Unit step response for $W_d(s)$ and $W_d^*(s)$.

$$W_d^*(s) = \frac{5s}{s^3 + 7.4s^2 + 21.16s + 25.6}. \qquad (7)$$

The unit step response for $W_d^*(s)$ is also shown in Figure 4. Its peak is 1/10th that of $W_d(s)$. The differential equation associated with $W_d(s)$ is

$$\dddot{\theta} + 7.4\ddot{\theta} + 21.16\dot{\theta} + 25.6\theta = 50\dot{w}. \tag{8}$$

In words, $\theta(t)$ is not responding to the unit step, itself. Rather, it is responding to its derivative; which is a Dirac delta function having an intensity equal to 50. This is a big impulse! It is, in part, for this reason that the authors devote the entire section 3.5 (pp.142-152) on the effects of zeros and additional poles. A term s^m in the numerator of a transfer function corresponds to the mth derivative of the input. To further highlight such an influence, suppose that the input to (8) were a small amplitude sinusoid $\varepsilon\sin(\omega t)$. The derivative is $\varepsilon\omega\cos(\omega t)$. For a high frequency ω the amplitude could be yuge! (Thanks Bernie ☺).

Conclusion Aircraft transfer functions very often include zeros. Even if the controller contains no zeros, one must take care in requiring specifications that are based only on poles. Even if such specifications are achieved, it may well be that the actual dynamics are significantly different that those associated with the specified poles. □

We will now investigate how a different selection of the controller zeros might improve matters. Rather than outright guessing, we will first use a little trigonometry to guide the process.

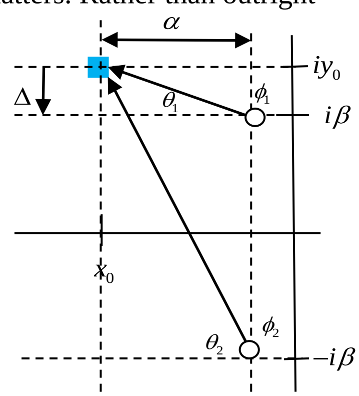


Figure 3 Geometry of the zeros.

The controller zeros must contribute $\phi_1 + \phi_2 = \phi$. From the figure at right, this is equivalent to $(180^o - \theta_1) + (180^o - \theta_2) = \phi$, which gives $\theta_1 + \theta_2 = 360^o - \phi \stackrel{\Delta}{=} \theta$. Hence, $\tan(\theta_1 + \theta_2) = \tan(\theta) \stackrel{\Delta}{=} \gamma$. Appealing to a tangent identity gives:

$\dfrac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \dfrac{(\Delta/\alpha) + (2\beta + \Delta)/\alpha}{1 - \Delta(2\beta + \Delta)/\alpha^2} = \gamma$. From this, we arrive at $\alpha^2 - \left[\dfrac{2(\beta+\Delta)}{\gamma}\right]\alpha - \Delta(2\beta + \Delta) = 0$.

The solution to this quadratic is: $\alpha_{1,2} = \dfrac{(\beta+\Delta)}{\gamma} \pm \sqrt{\left[\dfrac{(\beta+\Delta)}{\gamma}\right]^2 + \Delta(2\beta+\Delta)}$. Noting that $y_0 = \beta + \Delta$ gives:

$$\alpha_{1,2} = \frac{y_0}{\gamma} \pm \sqrt{\left(\frac{y_0}{\gamma}\right)^2 + y_0^2 - \beta^2}. \tag{9}$$

Clearly, per Figure 3 we must have:

$$\alpha = \frac{y_0}{\gamma} + \sqrt{\left(\frac{y_0}{\gamma}\right)^2 + y_0^2 - \beta^2} \quad \text{for } \Delta \ge 0. \tag{10a}$$

Carrying out the same procedure for $\Delta < 0$ also results in (9). Hence, we can conclude that:

$$\alpha = \frac{y_0}{\gamma} - \sqrt{\left(\frac{y_0}{\gamma}\right)^2 + y_0^2 - \beta^2} \quad \text{for } \Delta < 0. \tag{10b}$$

The resulting controller zeros are:

$$z_{1,2} = (-x_0 + \alpha) \pm i\beta. \tag{11}$$

For the specified closed loop pole $r = x_0 + iy_0$, the root locus magnitude criterion gives

$$K = 1/\left|G_c(r)G_p(r)\right|. \tag{12}$$

And so, for given r and ϕ we can choose various values for Δ, and view the resulting closed loop step response.

It took only a few guesses for Δ to arrive at what I believe is perhaps the best value: $\Delta = -0.1$. The root locus, and resulting closed loop FRF for $K = 1.5$ are shown below. Notice that the third real root is not shown in the root locus, as it its time constant is smaller than $\tau = 1/12$.

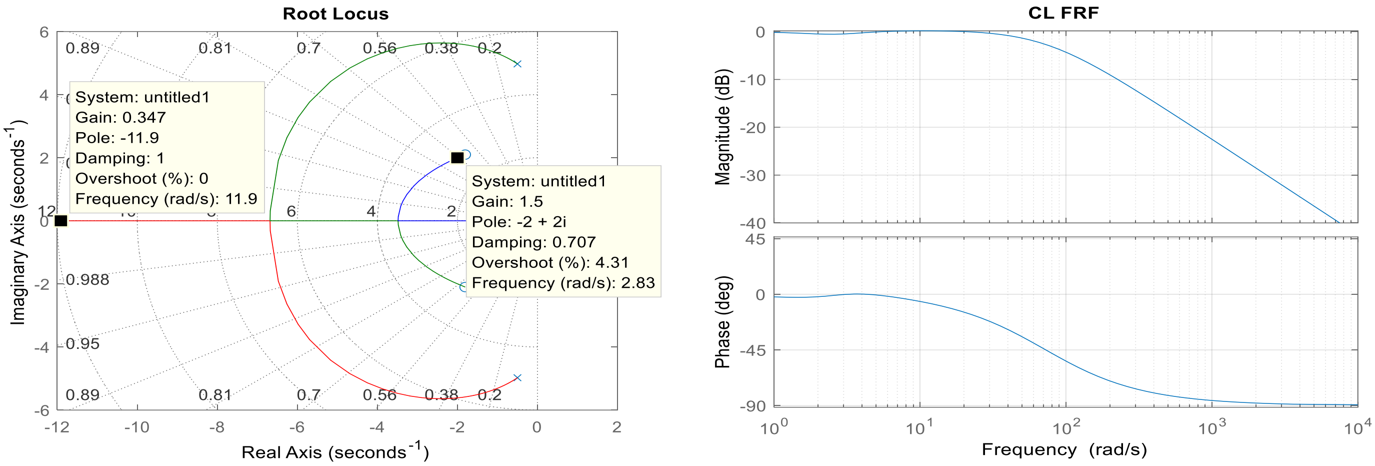


Figure 10 Closed loop root locus, step response, and FRF.

The controller transfer function is: $G_c(s) = \dfrac{1.502s^2 + 5.426s + 11.53}{s}$. The behavior of the root locus in Figure 10 is distinctly different than it is in Figure 2.

The closed loop system command and disturbance step responses are shown below.

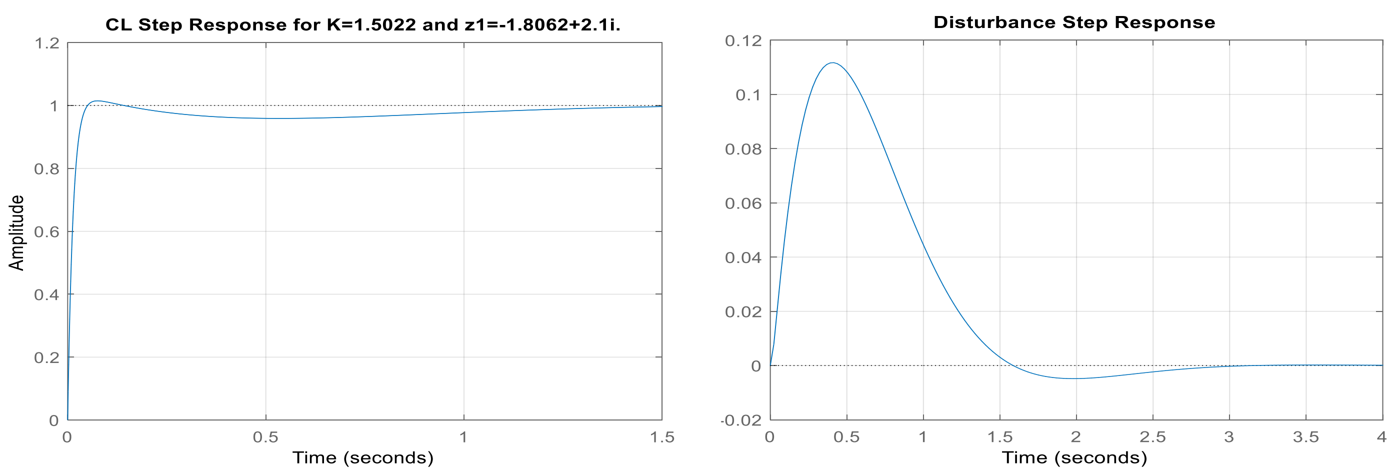


Figure 11 Closed loop system command (LEFT) and disturbance (RIGHT) step response.

The command system step response in Figure 11 is a significant improvement over that in Figure 3. The disturbance response shown at right is also a vast improvement over that in Figure 4.

Conclusion

I, myself, was stunned by the level of improvement in the command step response. The calculations (9-12) allowed me to investigate a variety of proposed controller zeros. Most gave, at best, the same responses as those in Figures 3 and 4. It was only when I ever so slightly perturbed them by $\Delta = -0.1$ that I discovered Figures 11 and 12.

It should be emphasized that <u>I did not use any more knowledge</u> than you, yourselves, have at this point in the course. What it took to achieve this result was simply curiosity, a trigonometric identity, and a couple of Saturday hours to get things straightened out. Over my dozens of years teaching this material I have found that, invariably, each semester I learn something new. The above discovery is well worth adding to my list. I suppose that I am destined to remain a student. ☺

Matlab Code for Example 2

```
%PROGRAM NAME: lec4ex2.m
Gp=tf(50,[1 1 25]);
x0=-2; y0=2;   %CL pole location
phi=300;   %Need PID conjugate zeros to give 300 degrees
g=tand(360-phi);
%del=distance from y0 to y-coord. of zero
%del >= 0: y-coord of zero < y0
%del<0: y-coord  of zero > y0
delmin=y0*(1-sqrt(1+1/g))
del=-0.1
b=y0-del; %Choose y-coordinate of upper zero
if del<0
    a=y0/g - sqrt((y0/g)^2 + y0^2 - b^2);
else
    a=y0/g + sqrt((y0/g)^2 + y0^2 - b^2);
end
z1=x0+a + 1i*b
z2=conj(z1);
s=tf('s');
Gc=(s-z1)*(s-z2)/s;
figure(1)
rlocus(Gc*Gp)
grid
%Find necessary gain:
p=x0+1i*y0;
[n,d]=tfdata(Gp,'v');
m=length(d);
c1=0; d1=0;
for k=1:m
    c1=c1+n(k)*p^(m-k);
    d1=d1+d(k)*p^(m-k);
end
MGp=abs(c1/d1);
[n,d]=tfdata(Gc,'v');
m=length(d);
c1=0; d1=0;
for k=1:m
    c1=c1+n(k)*p^(m-k);
    d1=d1+d(k)*p^(m-k);
end
MGc=abs(c1/d1);
K=(MGp*MGc)^-1
%--------------------
Gc=K*Gc;
W=feedback(Gc*Gp,1);
figure(3)
step(W)
title(['CL Step Response for K=',num2str(K),' and z1=',num2str(z1),'.'])
grid
figure(4)
bode(W)
title('CL FRF')
grid
%-----------------------------
%Disturbance Response:
Wd=feedback(Gp,Gc);
figure(4)
step(Wd)
title('Disturbance Step Response')
grid
```