Lecture 24 Replacing a Continuous-Time Controller with a Discrete-Time Controller (4/19/20)

We now have all the machinery to address the problem that might be of greatest interest to some students. Prior to addressing sampled systems, we spent a lot of time designing analog controllers. Hopefully, you are comfortable with the methods of (i) equating coefficients' and (ii) root-locus based pole placement for this purpose. In this lecture we will address examples, wherein it is desired to replace $G_c(s)$ by $G_c(z)$.

Example 1 This example concerns PROBLEM 3 of HOMEWORK 2. The beginning of that problem states the following:

The Root Locus-based *pole placement* method was used to design a unity-feedback control system for the plant $G_p(s)=\dfrac{10}{s(s+2)}$. The result was a *lead controller* $G_c(s)=\dfrac{4.13(s+2.75)}{s+9.55}$. The resulting OL and CL transfer functions are:

$$G(s)=G_c(s)G_p(s)=\frac{41.3(s+2.75)}{s(s+2)(s+9.55)} \quad \text{and} \quad W(s)=\frac{41.3s+113.57}{s^3+11.55s^2+60.4s+113.57}.$$

In this example, the goal is to replace $G_c(s)=\dfrac{4.13(s+2.75)}{s+9.55}$ by a suitable $G_c(z)$.

Choosing the Sampling Time

Selecting the appropriate sampling time requires that one have knowledge of the CL system BW. The CL Bode plot is given at right. We see that the CL BW is $\omega_{-3dB}=6.75\,r/s$. We also see that the magnitude is -40dB at $\omega_{-40dB}=64.2\,r/s$. If we choose $\omega_N=65\,r/s$, then $T=\pi/\omega_N=0.0483\,s$. Since I prefer round numbers, I will choose $T=0.05\,s$

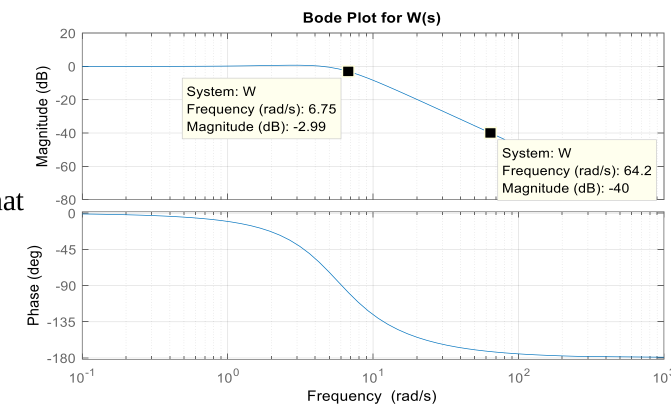

Figure E1.1 CL Bode plot.

Choosing the controller conversion method

Here, I will choose the 'zoh' method.

The Bode plots for $G_c(s)$ and $G_c(z)$ are shown at right. They agree reasonably well. However, I am personally not impressed.

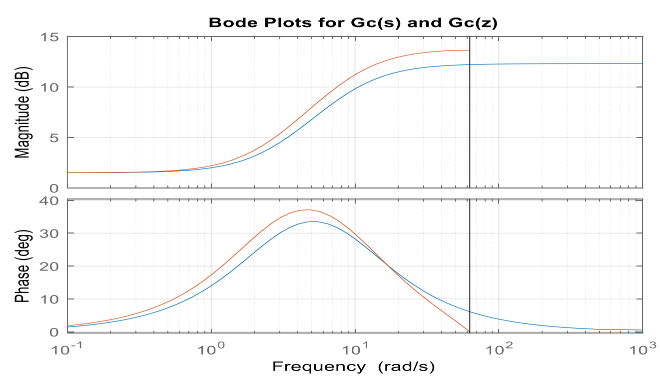

Figure E1.2 Bode plots for $G_c(s)$ and $G_c(z)$.

Constructing $ZOH(s)$ ***and the periodic version of*** $G_c(z)$

```
s=tf('s');
z1=exp(-s*T);
ZOH=(1-z1)/(T*s); %ZOH TF
[nc,dc]=tfdata(Gcz,'v');
GGcz=(nc(1)+nc(2)*z1)/(dc(1)+dc(2)*z1); %Periodic Gc(z)
```

Computing the Closed Loop TF

```
Q=GGcz*ZOH*Gp;
What=feedback(Q,1);
```

Zoomed Bode plots for $W(s)$ and $\hat{W}(s)$ are shown at right. With the exception of the phase drop above $\omega = 10\,r/s$ they compare reasonably well.

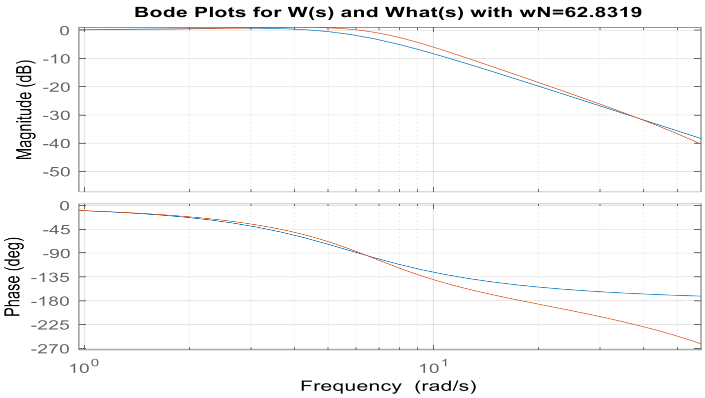

Figure E1.3 Zoomed Bode plots for $W(s)$ and $\hat{W}(s)$.

Validation via Comparison of Impulse and Step Responses

To compute these responses is not as easy as one might expect it to be. Matlab will not compute the <u>impulse</u> response for transfer functions that have time delay elements. In our case, both the ZOH and the controller have such elements. We therefor need to approximate the time delay element z1 using a Pade expansion:

```
z1p=pade(z1,3);  %I chose a 3rd order expansion because a little bird told me to ☺
ZOHp=(1-z1p)/(T*s);
GGcz=(nc(1)+nc(2)*z1p)/(dc(1)+dc(2)*z1p);
Qp=GGcz*ZOHp*Gp;
Whatp=feedback(Qp,1);
figure(13)
impulse(W,Whatp);
title('Impulse Responses for W(s) and What(s)')
legend('W','What')
grid
```

From the figures at right, we see that both the impulse and step responses compare reasonably well. □

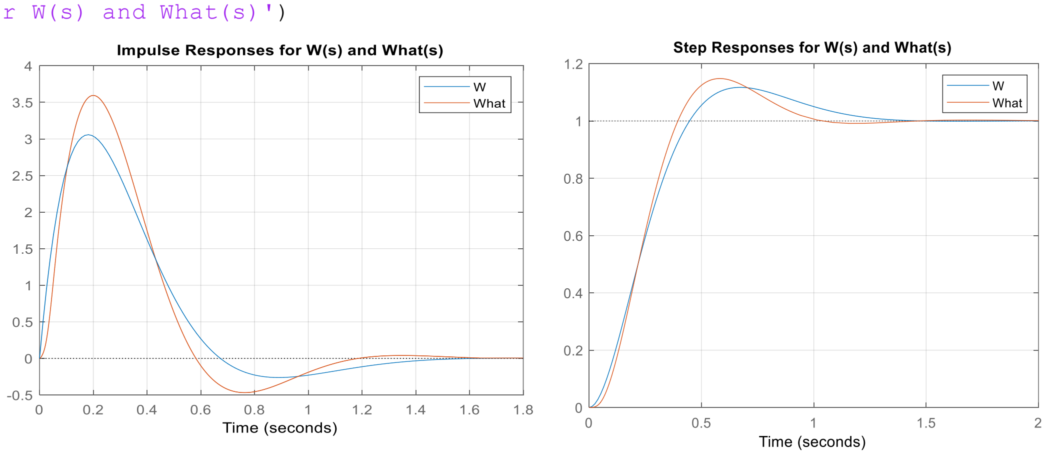

Figure E1.4 Comparison of impulse (LEFT) and step (RIGHT) responses.

Example 2 In this example we return to PROBLEM 4 in HOMEWORK 2. It states: The position/torque transfer function for a robotic arm is $\frac{Y(s)}{T(s)} \overset{\Delta}{=} G_p(s) = \frac{10}{s^2 + 0.2s + 100}$. It is desired to design a unity feedback control system using a PD controller $G_c(s) = K_p + K_d s$. The designed controller was $G_c(s) = 190 + 8.924s$. The goal of this example is to replace it by a suitable digital controller $G_c(z)$

Choosing the Sampling Time

From the CL Bode plot is given at right, we see that the CL BW is $\omega_{-3dB} = 118 r/s$. We also $\omega_{-20dB} = 10^3 r/s$. Since the roll-off rate is 20dB/decade, we have $\omega_{-40dB} = 10^4 r/s$. Taking this to be ω_N results in $T = \pi / 10^4$. We will use this value.
If we choose
, then $T = \pi / \omega_N = 0.0483 r/s$. Since I prefer round numbers, I will choose $T = 0.05 r/s$

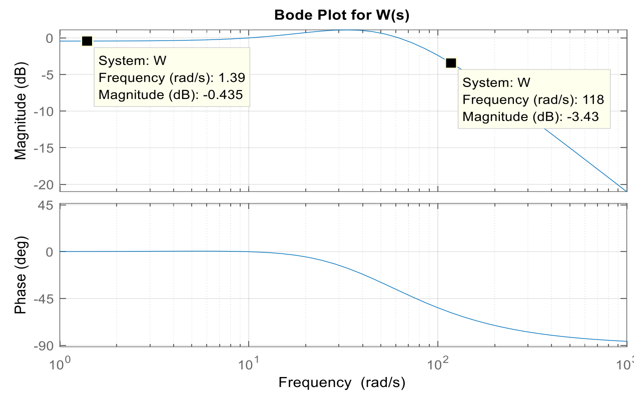

Figure E2.1 CL Bode plot.

Choosing the controller conversion method

Since $G_c(s)$ is not a 'proper' TF, we cannot use the 'zoh' method. Hence, we will try the 'tus' method. The Bode plots for $G_c(s)$ and $G_c(z)$ are shown at right. Overall, they agree very well. There is a very notable exception. Near the magnitude of $G_c(z)$ increases to 40dB above that of $G_c(s)$. This could be a problem.

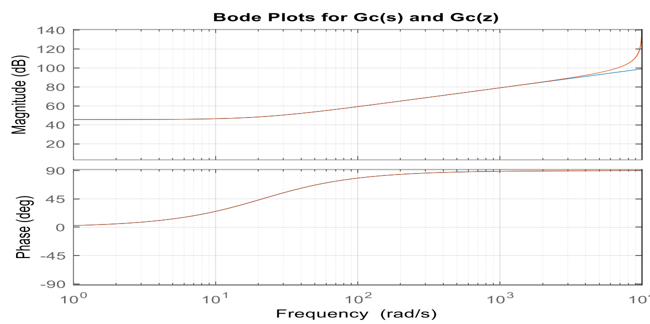

Figure E1.2 Bode plots for $G_c(s)$ and $G_c(z)$.

Constructing $ZOH(s)$ ***and the periodic version of*** $G_c(z)$.

This is the same as in Example 1.

Computing the Closed Loop TF

Zoomed Bode plots for $W(s)$ and $\hat{W}(s)$ are shown at right. For the most part, they compare well. Even so, there is a disturbing 40dB increase at ω_N. This is almost certainly due to the peak in $G_c(z)$. It goes up to the level of the static gain, and so could be a real problem.

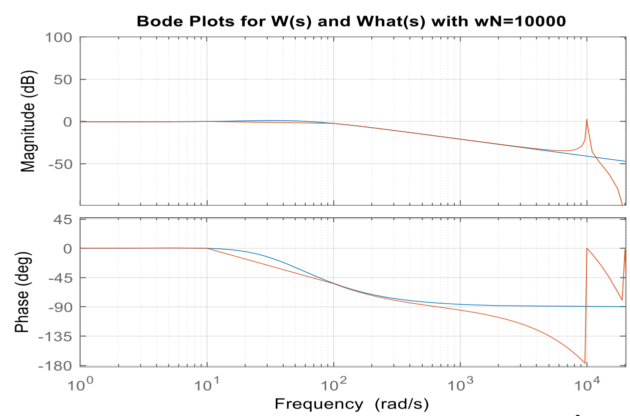

Figure E1.3 Zoomed Bode plots for $W(s)$ and $\hat{W}(s)$.

Validation via Comparison of Impulse and Step Responses

We do, indeed have a problem. ☹

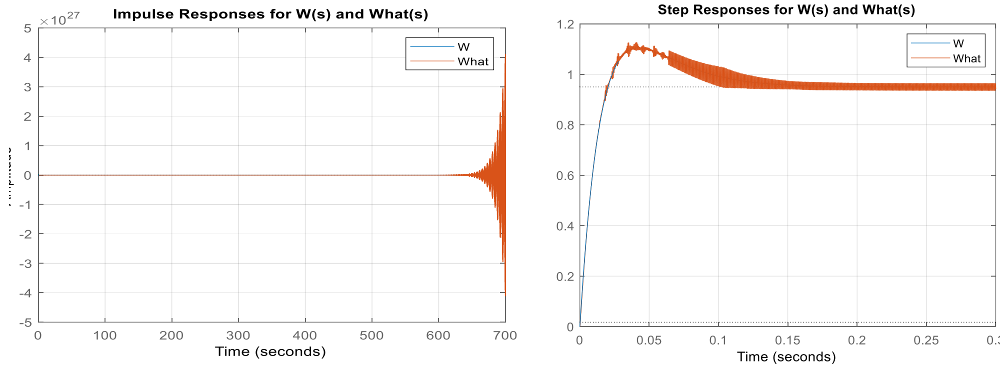

Figure E2.4 Comparison of impulse (LEFT) and step (RIGHT) responses.

The Problem with $G_c(z)$.

The discrete-time TF for the controller using the 'tus' flag is: $G_c(z) = 10^4\left(\dfrac{5.7z - 5.662}{z+1}\right)$ where $z = e^{sT} = e^{s\pi/\omega_N}$.

For $s = i\omega$, we have $z = e^{i\pi(\omega/\omega_N)}$. Consequently, for $\omega = \omega_N$ The denominator of $G_c(z)$ is zero. We can only conclude that for $G_c(s) = 190 + 8.924s$ the 'tus' method was a poor choice. The solution to this problem is to use a different method.

Write: $G(s) = K(s+a) = \dfrac{Y(s)}{U(s)}$. Then $\dot{u} + au = (1/K)y$. We can approximate $\dot{u} \cong \dfrac{u_k - u_{k-1}}{T}$. We then have

$\dfrac{u_k - u_{k-1}}{T} + au_k = (1/K)y_k$. This gives $(1+aT)u_k - u_{k-1} = (T/K)y_k$.

The z-transform is $(1+aT)U(z) - z^{-1}U(z) = (T/K)Y(z)$.

$$\frac{Y(z)}{U(z)} = G_c(z) = \frac{(1+aT) - z^{-1}}{(T/K)} = \frac{(1+aT)z - 1}{(T/K)z}.$$

For $K = K_d$ and $a = K_p / K_d$, we have $G_c(z) = 10^5\left(\dfrac{1.007z - 1}{3.52z}\right)$

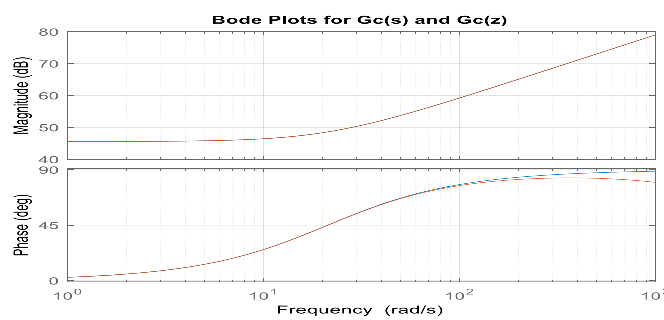

Figure E2.5 Bode plots of $G_c(s)$ and $G_c(z)$

The plots at right an below show that this controller does an excellent job.

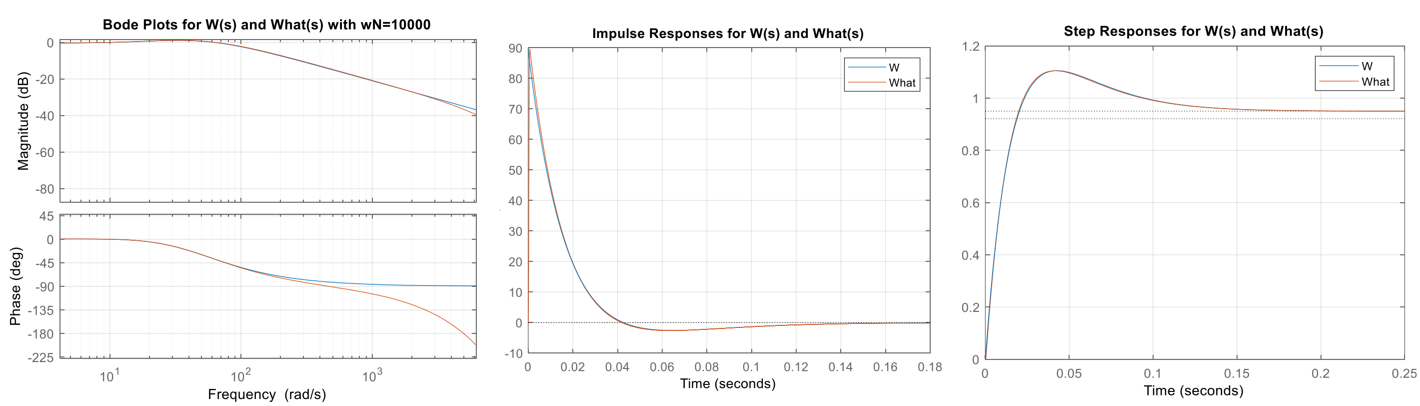

Figure E2.6 Bode plots, impulse responses, and step responses for $W(s)$ and $\hat{W}(z)$. □

Matlab Code

```
%PROGRAM NAME: lec24.m                (4/18/20)
%Example 1
%Choosing the appropriate value for T:
Gp = tf(10,[1 2 0]);
Gc = 4.13*tf([1 2.75],[1 9.55]);
G = Gc*Gp;
W = feedback(G,1);
figure(10)
bode(W)
title('Bode Plot for W(s)')
grid
T=0.05; %Chosen sampling period
wN=pi/T;
%--------------------------------
Gcz=c2d(Gc,T,'zoh');
figure(11)
bode(Gc,Gcz)
title('Bode Plots for Gc(s) and Gc(z)')
grid
%--------------------------------
s=tf('s');
z1=exp(-s*T);
ZOH=(1-z1)/(T*s); %ZOH TF
[nc,dc]=tfdata(Gcz,'v');
GGcz=(nc(1)+nc(2)*z1)/(dc(1)+dc(2)*z1); %Periodic Gc(z)
%-------------------------------
Q=GGcz*ZOH*Gp;
What=feedback(Q,1);
figure(12)
bode(W,What)
title(['Bode Plots for W(s) and What(s) with wN=',num2str(wN),''])
grid
%------------------------------
z1p=pade(z1,3);
ZOHp=(1-z1p)/(T*s);
GGcz=(nc(1)+nc(2)*z1p)/(dc(1)+dc(2)*z1p);
Qp=GGcz*ZOHp*Gp;
Whatp=feedback(Qp,1);
figure(13)
impulse(W,Whatp);
title('Impulse Responses for W(s) and What(s)')
legend('W','What')
grid
figure(14)
step(W,What);
title('Step Responses for W(s) and What(s)')
legend('W','What')
grid
%==========================================
%Example 2
Gp=tf(10,[1 .2 100]);
Kp=190; Kd=8.924;
Gc=tf([Kd Kp],1);
G=Gc*Gp;
W=feedback(G,1);
figure(20)
bode(W)
title('Bode Plot for W(s)')
grid
%---------------------
T=pi/10^4;
wN=pi/T;
%Gcz=c2d(Gc,T,'tus');
%Finite Difference Design:
K=Kd; a=Kp/Kd;
Gcz=tf([1+a*T,-1],[T/K,0],T);
figure(21)
bode(Gc,Gcz)
title('Bode Plots for Gc(s) and Gc(z)')
grid
%--------------------------------
```

```
s=tf('s');
z1=exp(-s*T);
ZOH=(1-z1)/(T*s); %ZOH TF
[nc,dc]=tfdata(Gcz,'v');
GGcz=(nc(1)+nc(2)*z1)/(dc(1)+dc(2)*z1); %Periodic Gc(z)
%-------------------------------
Ghat=GGcz*ZOH*Gp;
What=feedback(Ghat,1);
figure(22)
bode(W,What)
title(['Bode Plots for W(s) and What(s) with wN=',num2str(wN),''])
grid
figure(23)
bode(G,Ghat)
title(['Open Loop Bode plots with wN=',num2str(wN),''])
legend('G','Ghat')
grid
%------------------------------
z1p=pade(z1,3);
ZOHp=(1-z1p)/(T*s);
GGcz=(nc(1)+nc(2)*z1p)/(dc(1)+dc(2)*z1p);
Ghatp=GGcz*ZOHp*Gp;
Whatp=feedback(Ghatp,1);
figure(23)
impulse(W,Whatp);
title('Impulse Responses for W(s) and What(s)')
legend('W','What')
grid
figure(24)
step(W,What);
title('Step Responses for W(s) and What(s)')
legend('W','What')
grid
```