Lecture 21 Introduction to Discrete Time Systems and Their Analog Counterparts (4/15/20)

1. Motivation

Continuous-time (i.e. analog) systems are fundamental to understanding feedback control. However, rarely are analog controllers implemented to control analog plants. A discrete-time (i.e. digital) controller is a computer algorithm. Unlike an analog controller, a digital controller requires an analog-to-digital converter (A/D) to convert voltages into numbers, and a digital-to-analog converter (D/A) to convert numbers into voltages. However, the disadvantages of this additional hardware are offset by the advantages offered by the ease with which more advanced controllers can be implemented.

We will begin the motivation by revisiting some concepts that we have used throughout the course. Consider the term e^{-st} where $s = \sigma + i\omega$. This term is central to the definition of the Laplace Transform.

Definition 1.1. The Laplace transform of the time domain function $f(t)$ where $f(t) = 0$ for $t < 0$ [Note: For such a function the following is actually called the *one-sided* Laplace transform.] is defined as

$$F(s) \overset{\Delta}{=} \int_{t=0}^{\infty} f(t)e^{-st}dt \,. \tag{1.1}$$

This definition is not new. We have been using it throughout the entire semester. We present it here in order to delve a little deeper into its meaning. Substituting $s = \sigma + i\omega$ into (1.1) gives:

$$F(s = \sigma + i\omega) \overset{\Delta}{=} \int_{t=0}^{\infty} f(t)e^{-(\sigma + i\omega)t}dt \,. \tag{1.2}$$

Now, using the law of exponents, the quantity $f(t)e^{-(\sigma+i\omega)t}$ can be expressed as

$$f(t)e^{-(\sigma+i\omega)t} = [\, f(t)e^{-\sigma t}\,]e^{-i\omega t} \,. \tag{1.3}$$

Recall that in Cartesian coordinates a complex number is written as $x = x_R + ix_I$. In polar coordinates x is:

$$x = \sqrt{x_R^2 + x_I^2}\, e^{i\tan^{-1}(x_I/x_R)} = |\,x\,|e^{i\theta} \,. \tag{1.4}$$

where the term $e^{i\theta} = \cos\theta + i\sin\theta$ is Euler's identity. Clearly, from Pythagorean's Theorem, $|\,e^{i\theta}\,| = 1$. This highlights the reason that the quantity $e^{i\theta}$ is often referred to as the *unit-phasor*. It is a complex number with unit-magnitude and phase (i.e. angle) θ.

We are now in a position to take a closer look at (1.3). Specifically, it includes a real-valued term $f(t)e^{-\sigma t}$, multiplied by the unit-phasor $e^{i\theta} = e^{-i\omega t}$. Since $\theta = -\omega t$ is a function of time, we see that as $t \to \infty$, this term that is a vector in the complex plane spins clockwise, just as the second-hand on a clock does.

Demonstration: Run the following code to visualize how $e^{-i\omega t}$ behaves in polar and Cartesian coordinates as time progresses.

```
% Demonstration of e(-iwt)
T=1; %Chosen period of rotation
dt=T/30;
tmax=2*T;
t=0:dt:tmax;    nt=length(t);
w=2*pi/T;
x=exp(-1i*w*t);
xR=real(x);    xI=imag(x);
figure(1)
for k=1:nt
    pause(.2)
    subplot(2,1,1);plot([0,xR(k)],[0,xI(k)],'LineWidth',4)
    axis([-1.2,1.2,-1.2,1.2])
    title('exp(-i*w*t)')
xlabel('Real')
ylabel('Imag')
    grid
    tk=0:dt:t(k);
    subplot(2,1,2);plot(tk,[cos(w*tk);-sin(w*tk)]);
    axis([0,tmax,-1.2,1.2])
    xlabel('Time (sec)')
    legend('cos(w*t)','sin(w*t)')
    grid
end
```

Now that we see the component $e^{-i\omega t}$ behaves as $t \to \infty$, let's address the component $f(t)e^{-\sigma t}$. The first thing to note is that for $\sigma < 0$ the term $e^{-\sigma t} \to \infty$ as $t \to \infty$. As bad as this might seem, depending on how $f(t)$ behaves, it might not be as bad as it seems.

Example 1.1. Suppose we have $f(t) = e^{-at}$. Then $f(t)e^{-\sigma t} = e^{-(a+\sigma)t}$. Now we see that $f(t)e^{-\sigma t} \to \infty$ only when $a + \sigma < 0$; which is the same as the condition $\sigma < -a$. The question then is: Why should we care? The answer lies in the definition of the Laplace transform. If this condition holds, then the Laplace transform does not exist since the integral blows up. On the other hand, suppose that σ is chosen, such that the condition $\sigma > -a$. In this case, $f(t)e^{-\sigma t} \to 0$ as $t \to \infty$. Now, recall that the Laplace transform of $f(t) = e^{-at}$ is $F(s) = 1/(s+a)$. Viewed as a transfer function, we see that it has a pole at $s_1 = -a$. Hence, so long as we choose $s = \sigma + i\omega$ under the condition $\mathrm{Re}(s) > s_1$, the Laplace transform of $f(t) = e^{-at}$ exists. □

We will now see how the existence of the Laplace transform relates to the existence of the Fourier transform.

Definition 1.2 The Fourier transform of $f(t)$ having $f(t) = 0$ from $t < 0$ is simply $F(s = i\omega)$.

We have been using the Fourier transform throughout much of the semester. In fact, a Bode plot associated with a transfer function $F(s)$ is exactly the Fourier transform; albeit, plotted in dB and in log-frequency format.

Example 1.1 (continued) We will assume that $F(s) = 1/(s+a)$ is a stable transfer function. Then since it is well-defined for $\sigma > -a$, it should be clear that it is well-defined for $\sigma = 0$. However, the condition $\sigma = 0$ is exactly the condition $s = i\omega$. We can conclude that the Fourier transform $F(i\omega)$ is well-defined. Since we have viewed $F(s)$ as a transfer function, the Fourier transform $F(i\omega)$ is called the frequency response function (FRF). □

We can generalize the above example to arrive at the following fact.

Fact 1.1: The FRF of a system is only well-defined if the system is stable (i.e. has all poles in the LHP).

Note: One can show that the FRF is well defined if the system has poles on the *i*-axis. We will not pursue this fine point; except to say that a TF with poles at the origin does have a well-defined FRF. Even so, the area under the curve of the FRF magnitude is infinite.

Now at this point the astute student might ask: Then how is it that we have computed the Bode plots of unstable systems, and used them to arrive at stable closed loop systems?

Well, if you look carefully, all the open loop systems that we have constructed Bode plots for have been stable systems. We have never, for example, computed the Bode plot associated with $F(s)=1/(s-1)$. In view of the above, the Bode plot associated with this transfer function is not well-defined. The Bode plot computed by Matlab is shown at right.

From the looks of it, the transfer function as a static gain of 0dB and a -3dB BW of 1 rad/sec. Hence, from the Bode plot alone, one might readily conclude that this is a stable first order system. Well, it is a first order system, but it is not stable. When you give Matlab a TF in the Bode plot argument, Matlab does not check to see if it is a stable system. It assumes that you know what you're doing!

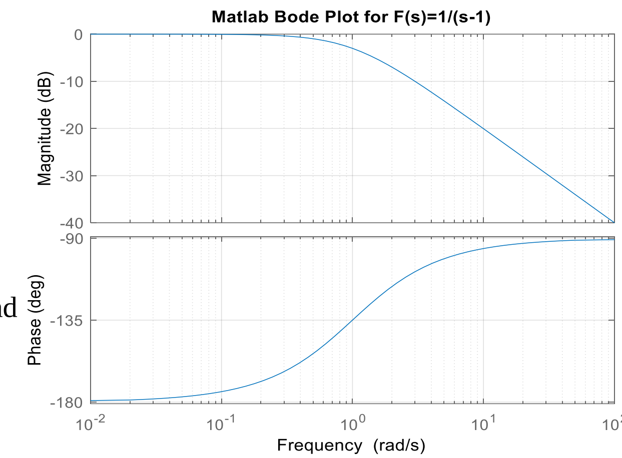

Figure 1. Bode plot for $F(s)=1/(s-1)$.

Since the importance of Fact 1.1 cannot be overstated, we summarize it as a:

WARNING: The Bode plot of an unstable system (i.e. having one or more poles in the proper RHP) is NOT well-defined, no matter what Matlab says.

REMINDER: I would not typically include the above material in lecture nots. Rather, I would present it at the board in class. It is not crucial to understand it in order to carry out homework problems. I included it because of the fact that I want the lecture notes to be as self-contained as possible. The drawback is that you will have to read through more material, should you be on the hunt for specifics.

2. Going from the Analog to the Digital World

For analog $f(t)$, let $f(kT)$ be its digital version, where T is a chosen sampling period (or interval). We begin this section by considering the digital version of $e^{-i\omega t}$, which is $e^{-i\omega(kT)}$. It should be pointed out that both $e^{-i\omega t}$ and $e^{-i\omega(kT)}$ are defined for all $\omega\in(-\infty,\infty)$. Even so, there is a major difference between them as functions of ω For any chosen ω, let $\omega'=\omega+2\pi m/T$, where m is any integer. Then:

$$e^{-i\omega'(kT)}=e^{-i(\omega+2\pi m/T)(kT)}=e^{-i\omega(kT)}e^{-i(2\pi m/T)(kT)}=e^{-i\omega(kT)}e^{-i(2\pi mk)}=e^{-i\omega(kT)}. \qquad (2.1)$$

The rightmost equality follows from the fact that both *m* and *k* are integers, and so $e^{-i(2\pi mk)} = 1$. Hence, we see that (2.1) is a periodic function of ω. The period is $\omega_s \stackrel{\Delta}{=} 2\pi / T$. This leads to the following

Fact 2.1 Let $x(t)$ be any function of time, and let $x(kT)$ be a sampled version of it. Then the frequency content of $x(kT)$ is uniquely defined only over the frequency region $\omega_N \le \omega < \omega_N$, where $\omega_N \stackrel{\Delta}{=} 0.5\omega_s$.

Definition 2.1 Since *T* is the sampling period, $\omega_s \stackrel{\Delta}{=} 2\pi / T$ is called the (radial) ***sampling frequency***. The frequency $\omega_N \stackrel{\Delta}{=} 0.5\omega_s$ is called the (radial) ***Nyquist frequency***.

Fact 2.1 is of such importance that we will restate it as

Fact 2.2 The frequency content of any discrete-time function $x(kT)$ is defined only up to the Nyquist frequency.

Now, the fact is that all data analysis is done digitally. Even though an accelerometer measure analog acceleration, it is digitized prior to using it in, for example, flight controllers. One must take care to choose a value for *T* such that the Nyquist frequency is *well above* the accelerometer BW. The question of how far above is *well above* does not have a single correct answer in the case of an accelerometer or just about any real system. An explanation for this follows from the following well-known theorem.

The Nyquist Sampling Theorem For any analog $f(t)$, let $F(i\omega)$ denote its Fourier transform. Then $f(t)$ can be perfectly recovered from its sampled version $f(kT)$ if and only if the following two conditions hold:

(C1): There exists an ω_0 such that $|F(i\omega_0)| = 0$ for all $\omega > \omega_0$, and

(C2): $\omega_s > 2\omega_0$.

Throughout the semester we have not encountered a single $f(t)$ that satisfies (C1).

Example 2.1 Consider $f(t)=e^{-t}$, with corresponding $F(s)=1/(s+1)$. Then $|F(i\omega)|=1/\sqrt{1+\omega^2}$. Clearly, there is no finite ω_0 such that (C1) holds. Let the Fourier transform of $f(kT)=e^{-kT}$ be denoted as $\hat{F}(i\omega)$. In words, view $\hat{F}(i\omega)$ as an estimate of $F(i\omega)$. The Bode plots at right illustrate how well $\hat{F}(i\omega)$ estimates $F(i\omega)$ when using a sampling frequency $\omega_s = 60\, rad/\sec$. The corresponding sampling period is $T = 2\pi/\omega_s = 2\pi/(2\omega_N) = \pi/\omega_N = \pi/30 = 0.1047\sec$.

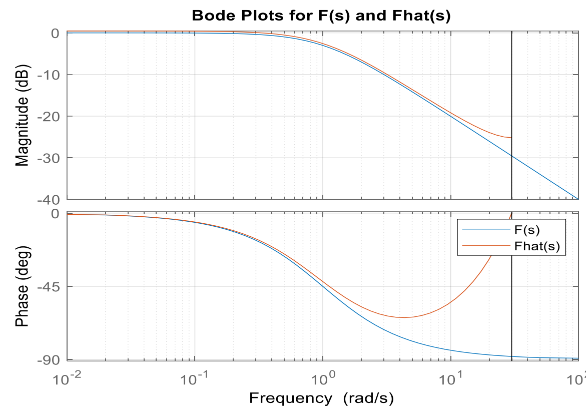

Figure 2.1 Bode plots of $F(i\omega)$ and $\hat{F}(i\omega)$.

The plot of $\hat{F}(i\omega)$ ends at the vertical black line that corresponds to the Nyquist frequency $\omega_N = 30\, rad/\sec$. The magnitude of $F(i\omega)$ is well-approximated by that of $\hat{F}(i\omega)$ for $\omega \le 10 rad/\sec$. The phase of $F(i\omega)$ is well-approximated by that of $\hat{F}(i\omega)$ for $\omega \le 1 rad/\sec$. Even though it is clear that $|F(i\omega_0)| \ne 0$ for all $\omega > 30 rad/\sec$, one could argue that it is no larger than 30dB below the static gain at those frequencies. Hence, it is a pretty good estimate. Others might argue that -30dB is not sufficiently small for their purposes. And others might argue that -20dB would be sufficiently small; in which case they would use $\omega_N = 20\, rad/\sec$.

```
%Example 2.1
s=tf('s');
F=1/(s+1);
figure(20)
bode(F);
wN=30; %Nyquist frequency
T=pi/wN;
Fhat=c2d(F,T,'impulse'); %Sampled system TF
figure(20)
bode(F,Fhat)
grid
title('Bode Plots for F(s) and Fhat(s)')    □
legend('F(s)','Fhat(s)')
```

The Mathematics of Going from Continuous to Discrete Time

A continuous time function $f(t)$ with $t \in [0,\infty)$ that is sampled every T seconds results is a discrete time function $f(kT)$, where $k \in \{0,1,\cdots,\infty\}$. The parameter T is called the sampling interval, or the sampling period. Its units (unless otherwise stated) are [seconds/sample]. The parameter $\omega_s = 2\pi/T$ is, therefore, the sampling frequency, with units [rad/sec].

Example 2.2 Consider $f(t)=e^{-at}$. Then $f(kT)=e^{-a(kT)}$. Define $\alpha \overset{\Delta}{=} e^{-aT}$. Then we can write $f(kT)=\alpha^k \overset{\Delta}{=} f(k)$, where the defined equality at right is often used for notational convenience. As simple as this may seem, there are a number of differences between $f(t)$ and $f(k)$.

<u>Difference #1</u>: By sampling, we no longer have any information about $f(t)$ at times other than the sample times. It could be doing all manner of crazy things!

<u>Difference #2</u>: The frequency structure of $f(t)$ extends over all frequencies $\omega \in (-\infty, \infty)$, whereas the frequency structure of $f(k)$ is defined only over $\omega \in (-\omega_N, \omega_N)$, where $\omega_N \overset{\Delta}{=} \omega_s / 2 = \pi / T$ is called the *Nyquist frequency*.

In Example 2.1 we used a Bode plot to visualize the differences between $F(i\omega)$ and $\widehat{F}(i\omega)$. In this example we will use algebra. For $f(t) = e^{-at}$, we have $F(s) = \dfrac{a}{s+a}$; consequently,

$$F(s) = \int_0^\infty e^{-at} e^{-st} dt = \frac{a}{s+a} . \tag{2.1}$$

We will now drive the expression for $\widehat{F}(i\omega)$. To this end, define the 'dummy' variable $z \overset{\Delta}{=} e^{sT}$. The Riemann sum approximation of the integral in
(2.1) is

$$\hat{F}(s) = \sum_{k=0}^{\infty} e^{-a(kT)} e^{-s(kT)} T = T \sum_{k=0}^{\infty} \left(e^{-aT}\right)^k \left(e^{-sT}\right)^k = T \sum_{k=0}^{\infty} \alpha^k z^{-k} = T \sum_{k=0}^{\infty} (\alpha z^{-1})^k . \tag{2.2}$$

Before we obtain an explicit form the rightmost infinite sum in (2.2), we need to point out the two 'dummy variables that were defined in it. The first is

$$\alpha \overset{\Delta}{=} e^{-aT} . \tag{2.3}$$

For a chosen sampling period T, (2.3) shows that the pole at $s_1 = -a$ has been transformed into $\alpha = e^{-aT}$. Now suppose that the system is stable. Then the pole $s_1 = -a$ is in the LHP. It should then be clear that $\alpha < 1$. The second dummy variable defined in (2.2) is

$$z \overset{\Delta}{=} e^{sT} . \tag{2.4}$$

Hence, for a chosen sampling period *T*, (2.4) shows how and value $s = \sigma + i\omega$ in the s-plane is mapped to a corresponding value in the *z*-plane. Let's look into the relation (2.4) between *s* and the dummy variable *z*. Write

$$z \overset{\Delta}{=} e^{sT} = e^{(\sigma + i\omega)T} = e^{\sigma T} e^{i\omega T} . \tag{2.5}$$

This is the polar form of the variable z. Its magnitude is $e^{\sigma T}$ and its phase is ωT. This highlights two properties about the mapping (2.4).

(P1): If $s = \sigma + i\omega$ is in the LHP, then $z = e^{sT}$ will be inside the ***unit circle*** (i.e. the subset of the complex plane where the values of the complex numbers have a magnitude less than on).

(P2): Consider the collection of imaginary numbers $\{s = i2\pi k\omega / T\}_{k=-\infty}^{\infty}$. This entire collection is mapped to a single point in the z-plane. To see what this point is, write

$$z = e^{(i2\pi k\omega/T)T} = e^{i2\pi k\omega} = e^{i\omega}. \tag{2.6}$$

Recall from Fact 2.1 that we must have $\omega_N \le \omega < \omega_N$. Hence, in words, all $2\pi k/T$ multiples of ω are mapped to the single frequency ω. The mapping (2.4) is shown visually below.

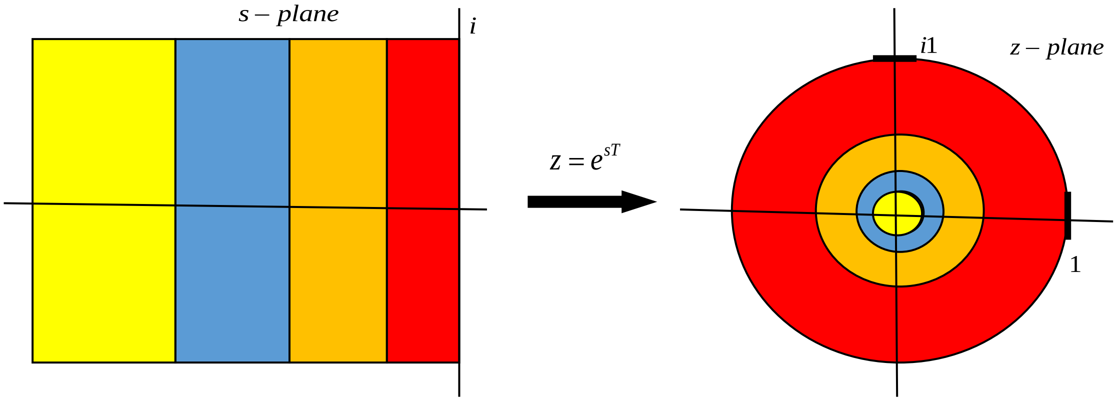

Figure 2.1 Qualitative illustration of the s-plane to z-plane mapping $z = e^{sT}$.

We now proceed to arrive at an explicit form for the rightmost infinite sum in (2.2). To this end, we will use the following very useful fact.

Fact 2.2 For any x consider the finite sum $\sum_{k=0}^{n} x^k$. Then if $|x|<1$, we have $\sum_{k=0}^{\infty} x^k = (1-x)^{-1}$.

<u>Proof</u>: The proof is based on the claim that $(1+x+x^2+\cdots+x^n)(1-x) = 1-x^{n+1}$. To see why this claim is true, notice that the left side is an $(n+1)^{\text{th}}$ degree polynomial. The value of the constant coefficient of this polynomial is clearly 1. With a little thought, one can see that the coefficient of the x term is zero. If one can see that, then with almost no more thought one can see that the coefficients of all the x^k terms are zero for $k \le n$. Finally, it should be clear that the coefficient of the term x^{n+1} is -1. Now suppose that $|x|<1$. Then $\lim_{n\to\infty} x^n = 0$, in which case we have $\left(\sum_{k=0}^{\infty} x^k\right)(1-x) = 1$. Right-multiplying both sides of this equation by $(1-x)^{-1}$ proves the above fact.

Remark 2.1 A proof of Fact 2.2 was given for two reasons. First, it is a very useful fact. Should one forget it, one need only write $(1+x+x^2+\cdots+x^n)(1-x)$, and go from there. Second, it holds for not only scalars, but also for matrices. Of course, if x is a matrix, the 1 is the identity matrix. In fact, Fact 2.2 is a very powerful result used in many fields of science and engineering.

Hence, if we assume in (2.2) that $|\alpha z^{-1}|<1$, we arrive at the explicit form

$$\hat{F}(s) = T\sum_{k=0}^{\infty} (\alpha z^{-1})^k = \frac{T}{1-\alpha z^{-1}} = \frac{Tz}{z-\alpha} \overset{\Delta}{=} F(z) \quad \text{where } \alpha \overset{\Delta}{=} e^{-aT} \tag{2.7}$$

The inside of the front cover of the boot includes a table of Laplace/z-transforms. One of the entries is:

$$Fs) = \frac{1}{s+a} \qquad f(t) = e^{-at} \qquad F(z) = \frac{z}{z - e^{-aT}}\,. \tag{2.8}$$

Remark 2.2 A comparison of (2.7) and $F(z)$ in (2.8) reveals that the latter does not include the factor of *T*. Even so, the Riemann sum approximation demands such a fact, as it approximates the term dt in the Laplace transform. Hence, were one to fail to include this factor in Overlaid Bode plots of $F(s)$ and $F(z)$, the magnitude of the latter would be shifted by *T*dB; making for a very confusing comparison. The reason that the table in the book, as well as tables in almost all textbooks do not include the factor of *T* is because more often than not, one is interested in obtaining the z-transform of the sampled impulse response; not a Bode plot comparison. The discrete approximation of the unit impulse $\delta(t)$ is $\hat{\delta}(kT) = 1/T$ for $k = 0$, and $\hat{\delta}(kT) = 0$ for $k \neq 0$. Hence, the sampled system impulse response will not include the factor of T in (2.7), as it will be cancelled by the $1/T$ associated with the approximate impulse. If you are a tad confused by this strange factor of T, do not be harsh on yourself. Many researchers, including faculty, who are not well-versed in this topic are also a tad confused. For this reason, throughout future developments, we will refer back to this fact, so that you might at least lessen any confusion.

Algebraic Comparison of $F(s = i\omega)$ ***and*** $F(z = e^{i\omega T})$

Clearly, for $F(s) = \dfrac{1}{s+a}$ we have:

$$F(i\omega) = \frac{1}{a + i\omega} = M(\omega)e^{i\theta(\omega)} \quad \text{where} \quad M(\omega) = \frac{1}{\sqrt{a^2 + \omega^2}} \quad \text{and} \quad \theta(\omega) = -\tan^{-1}(\omega / a)\,. \tag{2.9}$$

We have been using (2.9) throughout the better part of the semester. From (2.7) we have:

$$\hat{F}(s) = F(z = e^{i\omega T}) = \frac{T}{1 - \alpha e^{-i\omega T}} = \frac{T}{[1 - \alpha\cos(\omega T)] + i\alpha\sin(\omega T)}\,. \tag{2.10a}$$

The magnitude and phase of (2.10) are

$$\widehat{M}(\omega) = \frac{T}{\sqrt{[1 - \alpha\cos(\omega T)]^2 + [\alpha\sin(\omega T)]^2}} = \frac{T}{\sqrt{1 + \alpha^2 - 2\alpha\cos(\omega T)}}\,. \tag{210b}$$

and

$$\widehat{\theta}(i\omega) = -\tan^{-1}\{\alpha\sin(\omega T)/[1 - \alpha\cos(\omega T)]\}\,. \tag{2.10c}$$

To illustrate these differences graphically, let $a = 1$, and let $\omega_s = 60 rad/\sec.$. We will compute and overlay (2.9) and (2.10) over the frequency range $\omega \in [0.01, 100]$.

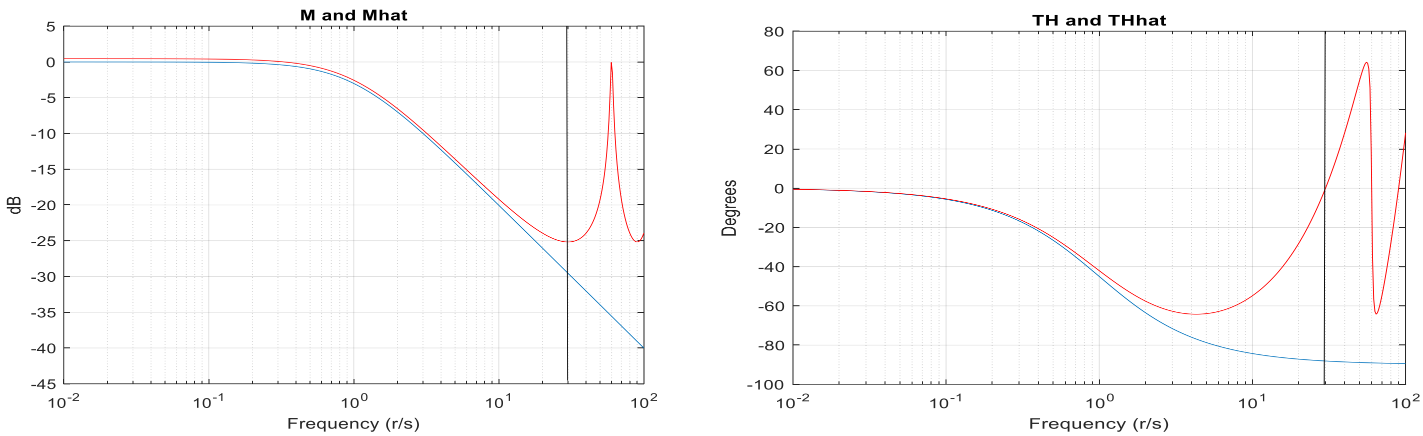

Figure 2.2 Comparison of magnitudes (LEFT) and phases (RIGHT) related to $F(i\omega)$ and $\hat{F}(i\omega)$ for $a=1$, and let $\omega_s = 60$.

The plots in Figure 2.2 are the same as those in Figure 2.1, with one exception. When using the Bode command, Matlab will plot the sampled system FRF only up to $\omega_N = 60$. Having the algebraic expressions in (2.10) allowed use to plot beyond the Nyquist frequency. The reason for doing this is to illustrate the periodic nature of the sampled system FRF. The reason that is does not look periodic in Figure 2.2 is because of the log nature of the frequency axis. Plots in linear frequency are given below.

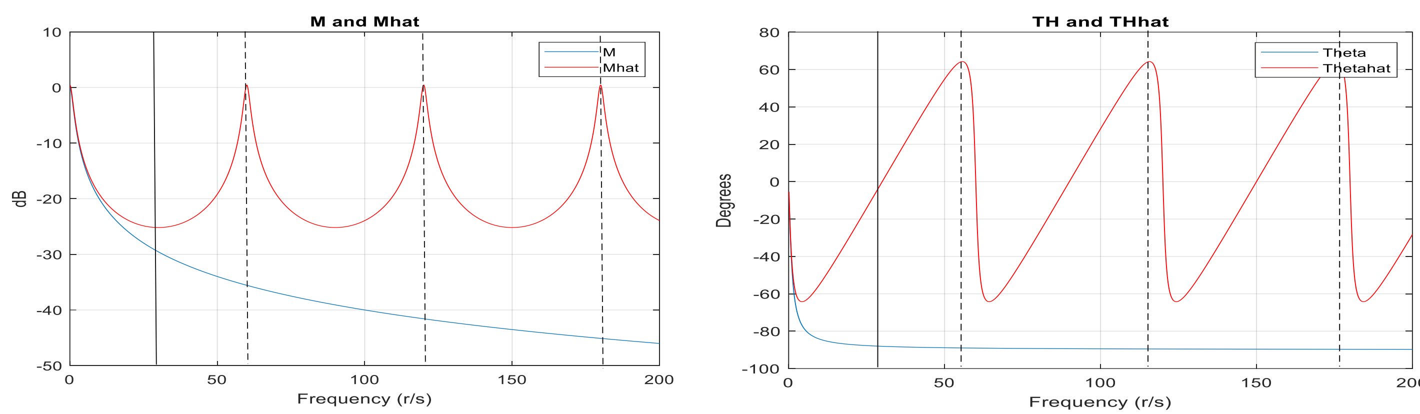

Figure 2.3 Plot of the curves in Figure 2.2, only using a linear frequency axis.

Clearly, the magnitude and phase expression in (2.10) are periodic with period $\omega_s = 60$. □

We will complete this lecture with the standard definition of the z-transform of a sequence $\{f_k\}_{k=0}^{\infty}$.

Definition 2.2 The z-transform of $\{f_k\}_{k=0}^{\infty}$ is $F(z) = \sum_{k=0}^{\infty} f_k z^{-k}$. Note that this standard definition does not include a factor of T. One reason is that the index *k* need not be related to time. For another reason, see Remark 2.2.

Matlab Code

```
%PROGRAM NAME: lec21.m
% Demonstration of e(iwt)
T=1; %Chosen period of rotation
dt=T/30;
tmax=2*T;
t=0:dt:tmax;    nt=length(t);
w=2*pi/T;
x=exp(-1i*w*t);
xR=real(x);    xI=imag(x);
figure(1)
for k=1:nt
    pause(.2)
    subplot(2,1,1);plot([0,xR(k)],[0,xI(k)],'LineWidth',4)
    axis([-1.2,1.2,-1.2,1.2])
    title('exp(-i*w*t)')
xlabel('Real')
ylabel('Imag')
    grid
    tk=0:dt:t(k);
    subplot(2,1,2);plot(tk,[cos(w*tk);-sin(w*tk)]);
    axis([0,tmax,-1.2,1.2])
    xlabel('Time (sec)')
    legend('cos(w*t)','sin(w*t)')
    grid
end
%===================================
%Example 2.1
s=tf('s');
F=1/(s+1);
figure(20)
bode(F);
wN=30; %Nyquist frequency
T=pi/wN;
Fhat=c2d(F,T); %Sampled system TF
figure(20)
bode(F,Fhat)
grid
title('Bode Plots for F(s) and Fhat(s)')
legend('F(s)','Fhat(s)')
%====================================
%Example 2.2
%Example 1
a=1; ws=60; T=2*pi/ws; aa=exp(-a*T);
w=logspace(-2,2,500);
F=a*(a+1i*w).^-1;
MdB=20*log10(abs(F));
TH=angle(F)*(180/pi);
Fhat=T*(1-aa*cos(w*T) +1i*aa*sin(w*T)).^-1;
MhatdB=20*log10(abs(Fhat));
THhat=angle(Fhat)*(180/pi);
figure(30)
semilogx(w,MdB)
hold on
semilogx(w,MhatdB,'r')
title('M and Mhat')
xlabel('Frequency (r/s)')
ylabel('dB')
grid
figure(31)
semilogx(w,TH)
hold on
semilogx(w,THhat,'r')
title('TH and THhat')
grid
xlabel('Frequency (r/s)')
ylabel('Degrees')
%Replot in linear frequency:
w=0.1:0.1:200;
F=a*(a+1i*w).^-1;
MdB=20*log10(abs(F));
TH=angle(F)*(180/pi);
Fhat=T*(1-aa*cos(w*T) +1i*aa*sin(w*T)).^-1;
```

```
MhatdB=20*log10(abs(Fhat));
THhat=angle(Fhat)*(180/pi);
figure(32)
plot(w,MdB)
hold on
plot(w,MhatdB,'r')
title('M and Mhat')
xlabel('Frequency (r/s)')
ylabel('dB')
grid
legend('M','Mhat')
figure(33)
plot(w,TH)
hold on
plot(w,THhat,'r')
title('TH and THhat')
grid
xlabel('Frequency (r/s)')
ylabel('Degrees')
legend('Theta','Thetahat')
```