Lecture 20 Random Inputs, Outputs, and Noise (4/4/20)

In lecture 19 we motivated the topic of random processes by addressing turbulence and it influence on flight dynamics. In this set of notes we will construct the topic from the ground up, so to speak. Our construction will be carried out mainly in the frequency domain.

To begin, we will first define what a random process is. We will then restrict attention to a certain class of such processes. This will allows us to embark on a frequency domain description of such processes.

Definition 1. Let $X(t)$ be a function of time, where $t \in (-\infty, \infty)$. If for any chosen t we cannot perfectly predict what $X(t)$ is, then the quantity $X(t)$ is called a ***random variable***. The entire collection $\{X(t)\,;\, t \in (-\infty, \infty)\}$ is called a ***random process***.

In words, a random process is simply a time-indexed collection of random variables.

Notation: It is common to refer to a random process $\{X(t)\,;\, t \in (-\infty, \infty)\}$ simply as $X(t)$. This implicitly presumes that we have not chosen focus on any specific time, *t*. On the other hand, if the focus is on any particular time, then, as noted in Definition 1, $X(t)$ is a random variable.

Definition 2. For a random process $X(t)$, let $R_X(t,\tau) \stackrel{\Delta}{=} E[X(t)X(t+\tau)]$, where E(*) is the expectation operation. We will not delve into what this operation is in these notes. The function $R_X(t,\tau)$ is called the ***autocorrelation function*** associated with the random process $X(t)$.

In words, $R_X(t,\tau)$ may be viewed as what one would expect the value of the product $X(t)X(t+\tau)$ to be "on the average". We will restrict our attention to the following class of random processes.

Definition 3. A random process $X(t)$ whose autocorrelation function $R_X(t,\tau)$ can be expressed simply as $R_X(\tau)$ is called a ***wide sense stationary*** (*wss*) random process.

In words, a wss random process is one in which the correlation between any two random variables $X(t)$ and $X(t+\tau)$ depends only on how far apart they are separated in time (i.e. τ).

While the autocorrelation function $R_X(\tau)$ is commonly used by statisticians, engineers typically prefer to use the following.

Definition 4. Let $X(t)$ be a *wss* random process $X(t)$ with autocorrelation function $R_X(\tau)$. Then the function

$$S_X(\omega) \stackrel{\Delta}{=} \int_{-\infty}^{\infty} R_X(\tau)\, e^{-i\omega\tau}\, d\tau \qquad \text{(D4)}$$

is called the ***power spectral density*** (*psd*) function for $X(t)$. It is also called the ***Fourier transform*** of $R_X(\tau)$.

In contrast to $R_X(\tau)$ that describes the process via correlation in the time domain, $S_X(\omega)$ describe it in the frequency domain.

Example 1. Suppose that $R_X(\tau) = e^{-|\tau|}$. Since $R_X(5) = e^{-|5|} = 0.0067$, we can say that if any two random variables $X(t)$ and $X(t+\tau)$ are spaced more than 5 seconds apart, then they are uncorrelated with each other. In lay person language one might say they have nothing to do with each other. The *psd* is:

$$S_X(\omega) = \int_{-\infty}^{\infty} e^{-|\tau|} e^{-i\omega\tau} d\tau \cdot$$

Since this is not a calculus course, we will use a table of Fourier transform pairs to obtain this integral. For example, from

https://engineering.purdue.edu/~mikedz/ee301/FourierTransformTable.pdf we have the Fourier transform pair: $e^{-a|\tau|} \leftrightarrow \frac{2a}{a^2+\omega^2}$.

Hence: $$S_X(\omega) = \frac{2}{1+\omega^2} \cdot \qquad \text{(E1.1)}$$

This *psd* is plotted at right . We see that the process -3dB bandwidth (BW) is $\omega_{bw} = 1 rad / \sec$. We also see that the energy content has a high frequency roll-off rate of -20dB/decade. Hence, the shape of (E1.1) resembles that of the FRF of a first order system that has a pole at $-\omega_{bw} = -1$

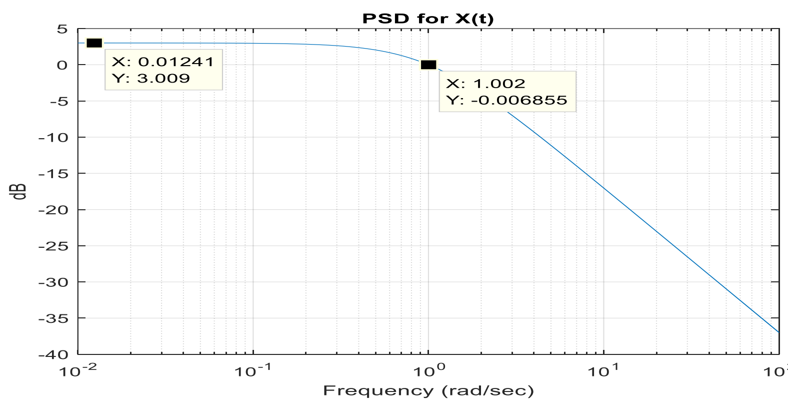


Figure E.1. The *psd* (E1.1).

Were $X(t)$ turbulence impacting the nose of a plane, it would be low frequency turbulence. Such turbulence would have sufficient energy to excite the phugoid mode, but not enough to notably excite the short period mode. □

The Concept of a Shaping Filter

This concept was discussed in the last lecture note. We will elaborate on it here. To this end, we need the following

Definition 5. Let $X(t)$ be a *wss* random process $X(t)$ with autocorrelation function $R_X(\tau) = \sigma_X^2 \delta(\tau)$ where $\delta(\tau)$ is the Dirac delta function. Then $X(t)$ is called a ***white noise process*** having intensity σ_X^2.

From (D4) it should be clear that the psd of a white noise process is $S_X(\omega) = \sigma_X^2$ for <u>every</u> $\omega \in (-\infty, \infty)$. Since the *psd* is the power per unit frequency, it follows that the total power (i.e. the area under the curve of the *psd*) is infinite. As such, a continuous-time white noise process does not exist!

The variance of a wss process is $R_X(0)$. Hence, a white noise process has variance $R_X(0) = \sigma_X^2 \times \infty$. This is why the parameter σ_X^2 for a white noise process is call the <u>intensity</u>. It is not the process variance. The variance is infinite.

Simulation of a continuous-time random processes process:

Clearly, a digital computer cannot simulate anything in truly continuous time. Nonetheless, such a process can be <u>approximately</u> simulated. To this end, consider the discrete-time random process $X(t_k = kT)$ where *T* is the sampling interval. Suppose that this is a white noise process. Then its variance is $R_X(0) = \sigma_X^2 / T$. Hence, as $T \to 0$, we see that $R_X(0) \to \sigma_X^2 \times \infty$. So for sufficiently small *T*, we will have a reasonable approximation to continuous-time white noise.

Before we embark on a simulation of this approximate white noise process, we need to address the influence of the sampling interval, *T*, on the frequency content of a wss process $X(t)$ having $R_X(\tau)$. The numerical approximation of (D4) is:

$$S_X(\omega) = \sum_{k=-\infty}^{\infty} R_X(kT) e^{-i\omega(kT)} T \,. \tag{1}$$

However, let's look carefully at the term $e^{-i\omega(kT)}$ that is defined for all $\omega \in (-\infty, \infty)$. For any chosen ω, let $\omega' = \omega + 2\pi m / T$, where m is any integer. Then:

$$e^{-i\omega'(kT)} = e^{-i(\omega + 2\pi m/T)(kT)} = e^{-i\omega(kT)} e^{-i(2\pi m/T)(kT)} = e^{-i\omega(kT)} e^{-i(2\pi mk)} = e^{-i\omega(kT)} \,. \tag{2}$$

The rightmost equality follows from the fact that both m and k are integers, and so $e^{-i(2\pi mk)} = 1$. Hence, we see that (2) is a periodic function of ω. The period is $\omega_s \stackrel{\Delta}{=} 2\pi / T$. This lead to the following

Fact 1.1 Let $x(t)$ be any function of time, and let $x(kT)$ be a sampled version of it. Then the frequency content of $x(kT)$ is defined only over the frequency region $\omega_N \le \omega < \omega_N$, where $\omega_N \stackrel{\Delta}{=} 0.5\omega_s$.

Definition 6. Since T is the sampling period, $\omega_s \stackrel{\Delta}{=} 2\pi / T$ is called the (radial) ***sampling frequency***. The frequency $\omega_N \stackrel{\Delta}{=} 0.5\omega_s$ is called the (radial) ***Nyquist frequency***.

Fact 1.1 is of such importance that we will restate it as

Fact 1. The frequency content of any discrete-time function $x(kT)$ is defined only up to the Nyquist frequency.

Now, the fact is that all data analysis is done digitally. Even though an accelerometer measure analog acceleration, it is digitized prior to using it in, for example, flight controllers. One must take care to choose a value for T such that the Nyquist frequency is well above the accelerometer BW.

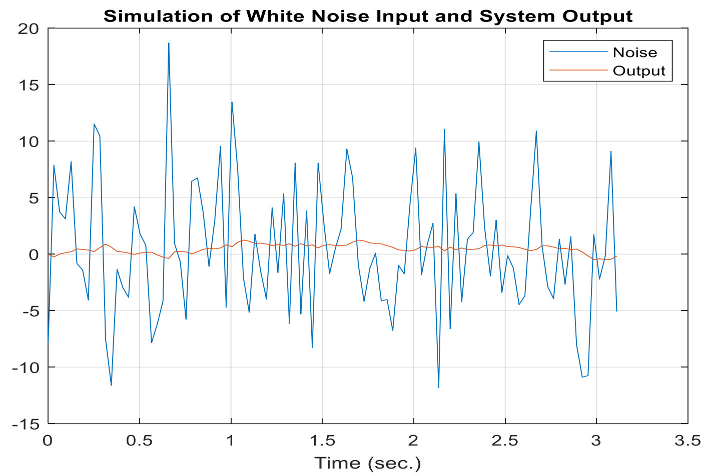
Example 2. In this example we will begin by simulating discrete-time white noise. We will then use it as the input to a first order system with transfer function $G(s) = 1/(s+1)$. Finally, we will investigate the structure of the output.

The first order of business is to arrive at a value for T. The system Bode plot at right shows that at frequencies above 100 rad/sec. the FRF magnitude is so small that it can be ignored. Hence, we will choose $\omega_N = 100\,rad/\sec$. This gives $\omega_s = 200\,rad/\sec$; which, in turn, gives $T = 2\pi/200 = 0.0314\sec$.

To simulate the white noise we will use the command 'normrnd(mu, sigma,n,1)' where the mean mu=0, the standard deviation sigma=1/sqrt(T), the number of rows is n, and the number of columns is 1.

Finally, we will use the 'lsim' command.

```
T=pi/100;
sigma=1/sqrt(T);
n=100;
x=normrnd(0,sigma,n,1);
%----------------------
G=tf(1,[1 1]);
%----------------------
t=0:n-1;
t=T*t';
y=lsim(G,x,t);
figure(20)
plot(t,[x,y])
```



The above code gives the figure at right. We see the noise is uncorrelated from point to point, while the output is clearly correlated. □

Figure E2. Plots of noise and the output.

Example 3. In this example we will run the discrete-time white noise through the transfer function $G(s) = 100/(s^2 + 2s + 100)$.

In the figure at right we see two differences in the output relative to Figure E2. The structure of the output is very oscillatory. There are approximately 4.5 oscillations over the course of 3.14 sec. The average oscillation period is therefore 0.7 sec. The corresponding radial frequency is about 9 rad/sec. This is close to the system resonance frequency of about 10 rad/sec. In words, the noise is exciting the low-damped resonance. □

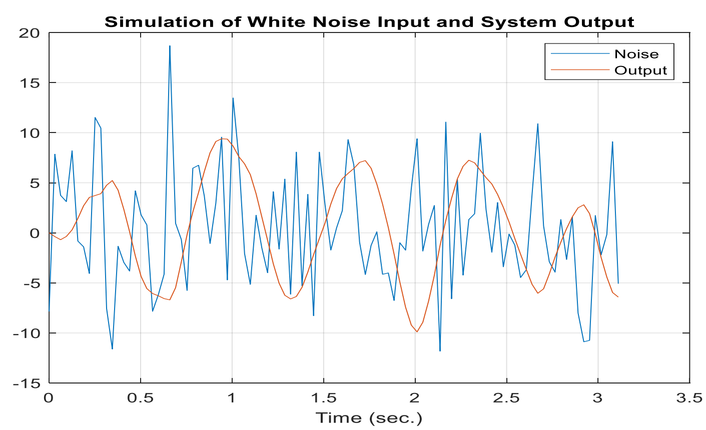


Figure E3. Plots of noise and the output.

We did not address the output related to the above examples directly in the frequency domain because there simply wasn't enough data to do so. We chose to use a small amount of data to illustrate the temporal properties of the output.

The *psd* of the output of a system excited by white noise:

In the last lecture we demonstrated the concept of what is commonly called a shaping filter. We will now elaborate on that concept.

Suppose that we excite a system with transfer function $G(s)$ with a random process $X(t)$. We know that in the frequency domain the output is $Y(i\omega) = G(i\omega)\hat{X}(i\omega)$, where $\hat{X}(i\omega)$ is the Fourier transform of $X(t)$. It follows that:

$$E[|Y(i\omega)|^2 = |G(i\omega)|^2 \; E[|\hat{X}(i\omega)|^2]. \qquad (3)$$

The quantity $E[|\hat{X}(i\omega)|^2]$ is the expected energy of $X(t)$ at the frequency ω. But that is exactly what the *psd* of $X(t)$ is. Since The input $X(t)$ is a random process, then so is the output. It then follows that $E[|Y(i\omega)|^2$ is the *psd* of the output. Consequently, (2) can be expressed as:

$$S_Y(\omega) = |G(i\omega)|^2 \; S_X(\omega). \qquad (4)$$

Now suppose that $X(t)$ is white noise with intensity σ_X^2. Recall that in this case $S_X(\omega) = \sigma_X^2$. Hence, (4) becomes:

$$S_Y(\omega) = \sigma_X^2 \; |G(i\omega)|^2. \qquad (5)$$

Hence, for a white noise input to a system with transfer function $G(s)$, the psd of the output has the shape of the squared magnitude of the FRF. If we view $G(s)$ as a filter, then it is, in fact, a ***shaping filter***. The psd of the output has the shape of the filter FRF magnitude-squared.

Matlab Code

```
%PROGRAM NAME: lec20.m   (4/2/20)
%Example 1
w=0.01:0.01:100;
S=2*(1+w.^2).^-1;
SdB=10*log10(S);
figure(10)
semilogx(w,SdB)
title('PSD for X(t)')
xlabel('Frequency (rad/sec)')
ylabel('dB')
grid
%Example 2
T=pi/100;
sigma=1/sqrt(T);
n=100;
x=normrnd(0,sigma,n,1);
%----------------------
G=tf(1,[1 1]);
%----------------------
t=0:n-1;
t=T*t';
y=lsim(G,x,t);
figure(20)
plot(t,[x,y])
title('Simulation of White Noise Input and System Output')
xlabel('Time (sec.)')
legend('Noise','Output')
grid
%Example 3
G=tf(100,[1 2 100]);
y=lsim(G,x,t);
figure(21)
plot(t,[x,y])
title('Simulation of White Noise Input and System Output')
xlabel('Time (sec.)')
legend('Noise','Output')
grid
```