Lecture 17 Feedback Control with Known Full State (3/3/20)

***The plant to be controlled*:**

$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$ (state equation) gives $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\,\mathbf{U}(s)$

$\mathbf{y} = \mathbf{Cx}$ (output equation) gives $\mathbf{Y}(s) = \mathbf{C}\,\mathbf{X}(s)$

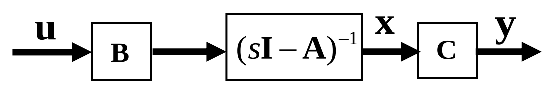


Figure 1 Plant block diagram.

***State feedback*:**

Regulator:

$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}[\mathbf{U}(s) - \mathbf{K}\mathbf{X}(s)]$ gives

$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}[\mathbf{U}(s) - \mathbf{K}\mathbf{X}(s)]$. This, in turn, gives

$[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})]\mathbf{X}(s) = \mathbf{B}\,\mathbf{U}(s)$. Let $\breve{\mathbf{A}} \overset{\Delta}{=} \mathbf{A} - \mathbf{BK}$. Then

$\mathbf{X}(s) = (s\mathbf{I} - \breve{\mathbf{A}})^{-1}\mathbf{B}\,\mathbf{U}(s)$. $\mathbf{Y}(s) = \mathbf{C}\,\mathbf{X}(s)$. (1a)

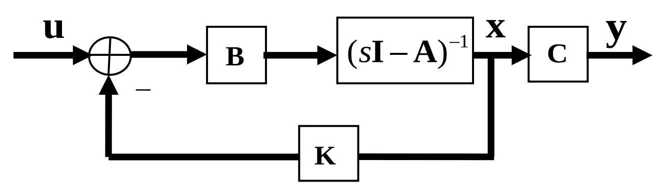


Figure 2(a) State feedback for regulator.

Command:

$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BK}[\mathbf{X}_r(s) - \mathbf{X}(s)]$ gives

$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{BK}[\mathbf{X}_r(s) - \mathbf{X}(s)]$. This, in turn, gives

$[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})]\mathbf{X}(s) = \mathbf{B}\mathbf{K}\mathbf{X}_r(s)$. Let $\breve{\mathbf{B}} \overset{\Delta}{=} \mathbf{BK}$

$\mathbf{X}(s) = (s\mathbf{I} - \breve{\mathbf{A}})^{-1}\breve{\mathbf{B}}\,\mathbf{X}_r(s)$. $\mathbf{Y}(s) = \mathbf{C}\,\mathbf{X}(s)$. (1b)

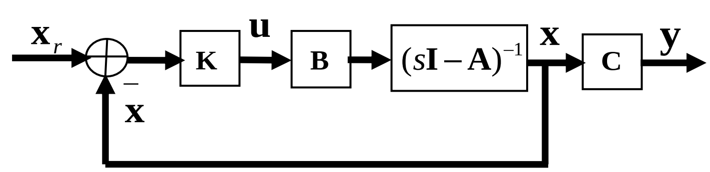


Figure 2(b) State feedback for command.

Remark 1. One must be careful to note that, unlike scalar-valued transfer functions, where the order of multiplication doesn't matter, for matrix-valued transfer functions it does. Specifically, while blocks precede other blocks in the block diagram, the latter blocks must be *right-multiplied* by the preceding blocks.

Remark 2. The stability of both the regulator system and the command system is dictated by the eigenvalues of $\breve{\mathbf{A}} \overset{\Delta}{=} \mathbf{A} - \mathbf{BK}$. These eigenvalues are the roots of the characteristic polynomial $p(s) = \det(s\mathbf{I} - \breve{\mathbf{A}})$.

Remark 3. The basis for obtaining the controller **K** is to specify *all* of the eigenvalues of $\breve{\mathbf{A}} \overset{\Delta}{=} \mathbf{A} - \mathbf{BK}$ (i.e. all of the CL poles). In this respect, this pole-placement method is an extension of the root locus-based placement of a single CL pole.

Remark 4. Figure 7.12 on p.487 is shown at right. As noted in the figure caption, it is the assumed system for the control law. This block diagram might seem strange to many students, as there is no command or disturbance input, as there are in Figures 1 and 2 above.

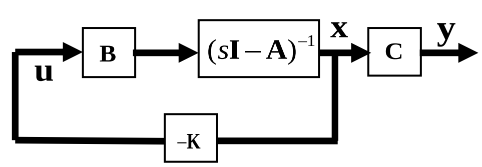


Figure 7.12 Assumed system for control law.

As such, it is not a proper *system*, since, by definition a system is a relation between an input and an output. A closer comparison of equations (1a) and (1b) reveals a single difference. In (1a) we have **B**; whereas in (1b) we have $\breve{\mathbf{B}} \overset{\Delta}{=} \mathbf{BK}$. The block diagram in Figure 7.12 gives:

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}(-\mathbf{BK})\,\mathbf{X}(s). \quad (2a)$$

This, in turn, gives: $$[s\mathbf{I}-(\mathbf{A}-\mathbf{BK})]\mathbf{X}(s)=(s\mathbf{I}-\breve{\mathbf{A}})\mathbf{X}(s)=\mathbf{0}. \tag{2b}$$

The term $s\mathbf{I}-\breve{\mathbf{A}}$ in (2b) is present in both (1a) and (1b). In other words, it doesn't matter if the controller is placed in the forward loop or the feedback loop. They have the same open loop transfer function that controls the closed loop stability.

***Example* 1.** Consider a plant with transfer function $G_p(s)=\dfrac{5}{s^2+s+25}\overset{\Delta}{=}\dfrac{Y(s)}{U(s)}$.

(a) Recover the differential equation, and then define the state vector $\mathbf{x}(t)=[x_1(t)\quad x_2(t)]^{tr}=[\dot{y}(t)\quad y(t)]^{tr}$. Derive the expressions for $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ related to the state equation $\dot{\mathbf{x}}=\mathbf{Ax}+\mathbf{B}u$ and the output equation $y=\mathbf{Cx}+\mathbf{D}u$.

<u>Solution</u>: $\ddot{y}+\dot{y}+25y=5u$ gives: $\dot{x}_1=-x_1-25x_2+5u$. Also: $\dot{x}_2=x_1$. So we can write:

$\dot{\mathbf{x}}=\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & -25\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}5\\ 0\end{bmatrix}u=\mathbf{Ax}+\mathbf{B}u$. Then $y=[0\quad 1]\mathbf{x}+0u=\mathbf{C}x+\mathbf{D}u$.

Hence, $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})=\left(\begin{bmatrix}-1 & -25\\ 1 & 0\end{bmatrix},\begin{bmatrix}5\\ 0\end{bmatrix},[1\quad 0],0\right)$.

(b) Use the 'acker' [**NOTE:** This command has been replaced by the command '**place**'.] command to find $\mathbf{K}$ that will place closed loop poles at $s_{1,2}=-10\pm i10$.

<u>Solution</u>: A=[-1 -25;1 0]; B=[5;0]; p=[-10+10*1i ; -10-10*1i]; K=acker(A,B,p) =[**3.8 35**]

(c) Verify the design by computing the eigenvalues of $\breve{\mathbf{A}}\overset{\Delta}{=}\mathbf{A}-\mathbf{BK}$.

<u>Solution</u>: AK=A-B*K; eigs(AK) = -10.0000 +/-10.0000i

(d) Use the ss2tf command to recover the *regulator* CL transfer function.

<u>Solution</u>: $\mathbf{X}(s)=(s\mathbf{I}-\breve{\mathbf{A}})^{-1}\mathbf{B}\,\mathbf{U}(s)$. $\mathbf{Y}(s)=\mathbf{C}\,\mathbf{X}(s)$

C=[0 1]; D=0; [num,den]=ss2tf(AK,B,C,D) num = [0 0 5] den = [1 20 200]

(e) Repeat (d) for the *command* CL transfer function. To this end, first compute the portion of $Y(s)$ due to input $Y_r(s)=X_2(s)$. Next, compute the portion of $Y(s)$ due to input $sY_r(s)=X_1(s)$. Finally, use superposition to obtain the total output $Y(s)$ as a function of $Y_r(s)$.

<u>Solution</u>: $\mathbf{X}(s)=(s\mathbf{I}-\breve{\mathbf{A}})^{-1}\breve{\mathbf{B}}\,\mathbf{X}_r(s)$.

BK=B*K; C2=eye(2); D2=zeros(2,2);

[numC denC]=ss2tf(AK,BK,C2,D2,**2**) numC = [0 175 0 ; 0 0 175] denC = [1 20 200]

In particular: $\dfrac{X_2(s)}{X_{r_2}(s)}=\dfrac{Y(s)}{Y_r(s)}=\dfrac{175}{s^2+20s+200}=W_{2,r_2}(s)$

[numC denC]=ss2tf(AK,BK,C2,D2,**1**) numC = [0 19 0 ; 0 0 19] denC = [1 20 200]

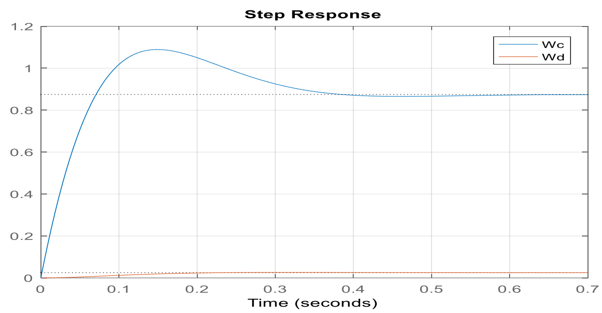
In particular: $\dfrac{X_2(s)}{X_{r_1}(s)} = \dfrac{Y(s)}{sY_r(s)} = \dfrac{19}{s^2+20s+200} = W_{2,r_1}(s)$.

Hence, by superposition: $Y(s) = W_{2,r_2}(s)\,X_{r_2}(s) + W_{2,r_1}(s)\,X_{r_1}(s) = \left(\dfrac{19s+175}{s^2+20s+200}\right)Y_r(s) = W(s)Y_r(s)$.

(f) Verify your answer in (e) by computing the unity feedback CL transfer function directly.

Solution: $G_p(s) = \dfrac{5}{s^2+s+25}$ and $G_c(s) = 3.8s+35$. So, $G_c(s)G_p(s) = \dfrac{19s+175}{s^2+s+25}$. Hence, $W(s) = \dfrac{19s+175}{s^2+20s+200}$.

Remark. As opposed to a command system, a regulator system is one where the input is viewed as a disturbance. In a well-designed regulator system, the output due to the disturbance will be minimal. Overlay the responses to a step command and to a step disturbance.
Solution: Wc=tf([19 175],[1 20 200]); Wd=tf(5,[1 20 200]);
step(Wc,Wd)   legend('Wc','Wd')   grid



***Example* 2.** In this example we will demonstrate how the 'acker' command works. Consider the second order underdamped plant: $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = c\,f(t)$. Let $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$ and $\mathbf{x}(t) = [x_1(t)\ x_2(t)]^{tr}$.

(a)(Give the state space 1[ST] order differential equation, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$.

Solution: $\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\omega_n^2 & -2\zeta\omega_n\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ c\end{bmatrix} f(t)$. [I have intentionally chosen a different form for **A**.]

(b) Let $\mathbf{u} = -\mathbf{K}\,\mathbf{x}$ (i.e. use a full-state feedback controller). Then we have $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x}$. Taking the Laplace transform gives $[s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})]\mathbf{X}(s) = \mathbf{0}$, and so the closed loop characteristic polynomial is:

$$p(s) = |s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})| = \left|\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \left(\begin{bmatrix}0 & 1\\ -\omega_n^2 & -2\zeta\omega_n\end{bmatrix} - \begin{bmatrix}0\\ c\end{bmatrix}\begin{bmatrix}K_1 & K_2\end{bmatrix}\right)\right|$$

$$= \left|\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \left(\begin{bmatrix}0 & 1\\ -\omega_n^2 & -2\zeta\omega_n\end{bmatrix} - \begin{bmatrix}0 & 0\\ cK_1 & cK_2\end{bmatrix}\right)\right| = \left|\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \left(\begin{bmatrix}0 & 1\\ -(\omega_n^2 + cK_1) & -(2\zeta\omega_n + cK_2)\end{bmatrix}\right)\right|.$$

$$= \left|\begin{bmatrix}s & -1\\ (\omega_n^2 + cK_1) & s + (2\zeta\omega_n + cK_2)\end{bmatrix}\right| = s[s + (2\zeta\omega_n + cK_2)] + (\omega_n^2 + cK_1)$$

Hence, $p(s) = s^2 + (2\zeta\omega_n + cK_2)s + (\omega_n^2 + cK_1) \overset{\Delta}{=} s^2 + \alpha_1 s + \alpha_2$. Now, suppose that we desire $p(s) = s^2 + 2\zeta'\omega_n' s + \omega_n'^2$ (i.e. pole placement). Then we can solve for the controller gains by setting:
$s^2 + (2\zeta\omega_n + cK_2)s + (\omega_n^2 + cK_1) = s^2 + 2\zeta'\omega_n' s + \omega_n'^2$. By equating coefficients, we end up with:

$$K_1 = (\omega_n'^2 - \omega_n^2)/c \text{ and } K_2 = 2(\zeta'\omega_n' - \zeta\omega_n)/c\,.$$

Numerical Values: Suppose that the plant has $c = 2$, and that the plant has $\zeta = 0.05$, and $\tau = 10\,\text{sec}$. Then $\zeta\omega_n = 1/\tau = 0.1\,r/s$ and $\omega_n = 0.1/\zeta = 20\,r/s$. Suppose that we want $\zeta' = 0.8$ and $\omega_n' = 30\,r/s$ [i.e. increased damping and BW] Then $\zeta'\omega_n' = 24 \Rightarrow \tau' = 1/24\ s$ [which leads to also faster response].

Then: $K_1 = (\omega_n'^2 - \omega_n^2)/c = (30^2 - 20^2)/2 = 250$ and $K_2 = 2(\zeta'\omega_n' - \zeta\omega_n)/c = 2(24 - 0.1)/2 = 23.9$.

Hence, $\mathbf{K} = [K_1 \quad K_2] = [250 \quad 23.9]$.

While this was straightforward, for higher order systems it becomes exponentially painstaking. Even so, the method is the same. It is called *Ackerman's* formula. The Matlab command is 'acker'.

$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -400 & -0.2 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. $\mathbf{s12} = \begin{bmatrix} -24 + i18 \\ -24 - i18 \end{bmatrix}$. acker(A,B,s12) = [250 23.9].

The general form of *Ackerman's* formula is given by (7.88) on p.492. □

***Example* 3.** In this example we will address the thermal system described by the transfer function (Example 7.9 on p.478):

$$G(s) = \frac{s+2}{s^2 + 7s + 12} = \frac{Y(s)}{U(s)}.$$

(a) Obtain the *controller* canonical form for (**A,B,C,D**).

Solution: Let $G_1(s) \stackrel{\Delta}{=} \frac{1}{s^2 + 7s + 12} = \frac{V(s)}{U(s)}$ and $G_2(s) \stackrel{\Delta}{=} s + 2 = \frac{Y(s)}{V(s)}$. Then $G(s) = G_1(s)G_2(s)$. The O.D.E. corresponding to $G_1(s)$ is: $\ddot{v} + 7\dot{v} + 12v = u$. Define the state $\mathbf{x} = [x_1 \quad x_2]^{tr} = [\dot{v} \quad v]^{tr}$. We then have:

$$\dot{\mathbf{x}} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad \text{and} \quad y = [1 \quad 2]\mathbf{x} + 0u = \mathbf{C}\mathbf{x} + \mathbf{D}u. \tag{1}$$

(b) Use Figure 7.10 on p.476 to obtain the *observer* canonical form for (**A,B,C,D**).

Solution: $$\dot{\mathbf{x}} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad \text{and} \quad y = [1 \quad 0]\mathbf{x} + 0u = \mathbf{C}\mathbf{x} + \mathbf{D}u. \tag{2}$$

(c) Clearly, the state variables are not composed of y and its derivatives. Suppose that we desire to compute the response of the system (**A,B,C,D**) to the initial condition $[\dot{y}_0 \quad y_0]^{tr}$. To this end, we need to find the state initial condition $\mathbf{x}_0$.

Solution:

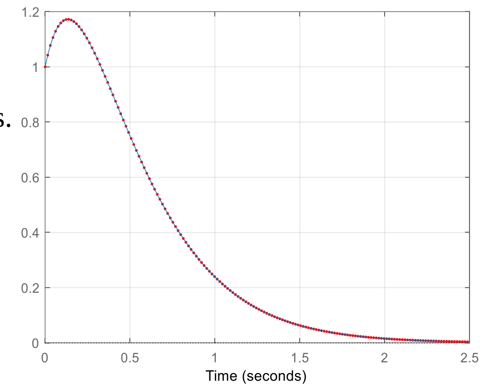
From the output equation in (1), we have: $y_0 = \mathbf{C}\mathbf{x}_0$. From this, we also have: $\dot{y}_0 = \mathbf{C}\dot{\mathbf{x}}_0$.

From the state equation in (1) we have: $\dot{\mathbf{x}}_0 = \mathbf{A}\mathbf{x}$, so that $\dot{y}_0 = \mathbf{C}\mathbf{A}\mathbf{x}_0$. These equations can be written as:

$\begin{bmatrix} \dot{y}_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \mathbf{CA} \\ \mathbf{C} \end{bmatrix}\mathbf{x}_0$. Hence, we arrive at: $\mathbf{x}_0 = \begin{bmatrix} \mathbf{CA} \\ \mathbf{C} \end{bmatrix}^{-1}\begin{bmatrix} \dot{y}_0 \\ y_0 \end{bmatrix}$. Specifically, $\mathbf{x}_0 = \begin{bmatrix} 1 & 6 \\ -0.5 & -2.5 \end{bmatrix}\begin{bmatrix} \dot{y}_0 \\ y_0 \end{bmatrix}$.

(d) For the initial condition $[\dot{y}_0 \quad y_0]^{tr} = [3 \quad 1]^{tr}$, use the 'initial' command to arrive at a plot of the system initial condition response.

Solution: Y0=[3 ; 1]; CAC=[C*A;C]; X0=CAC^-1*Y0; [y,t]=initial(sys,X0); plot(t,y) grid



(e) Solve for the initial response directly from the O.D.E. using Laplace transforms. Then overlay this response on your plot in (b).

Solution: $\ell(\ddot{y} + 7\dot{y} + 12y = 0) \Rightarrow [s^2Y(s) - sy_0 - \dot{y}_0] + 7[sY(s) - \dot{y}_0] + 12Y(s) = 0$.

$Y(s) = \dfrac{sy_0 + \dot{y}_0 + 7y_0}{s^2 + 7s + 12} = \dfrac{s+10}{s^2 + 7s + 12}$. YIC=tf([1 10],[1 7 12]); impulse(YIC)

The initial condition response is exactly that obtained in (d).

(f) Now consider the control law $u = -\mathbf{K}\mathbf{x}$. Then $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ becomes $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \overset{\Delta}{=} \breve{\mathbf{A}}\mathbf{x}$. Whereas, the plant poles are the eigenvalues of **A**, the CL poles are the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K} \overset{\Delta}{=} \breve{\mathbf{A}}$. For each of the representations in (a-b), use the 'acker' command to find the **K** that will place CL poles at $p_{1,2} = -3 \pm 0.5i$.

Solution:

Ac=[-7 -12;1 0]; Bc=[1;0]; Kc=acker(Ac,Bc,p) = [-1.000 -2.750]

Ao=[-7 1;-12 0]; Bo=[1;2]; Ko=acker(Ao,Bo,p) = [-1.750 0.375]

(g) Use the ss2tf command to recover the CL *regulator* (i.e. with the controller in the feedback loop) transfer function for the *canonical* and *observable* controller forms.

Solution:

AAc=Ac-Bc*Kc; [n,d]=ss2tf(AAc,Bc,Cc,0) n =[1 2] & d =[1 6 9.25]. So, $W(s) = \dfrac{s+2}{s^2 + 6s + 9.25}$.

AAo=Ao-Bo*Ko; [n,d]=ss2tf(AAo,Bo,Co,0) n =[1 2] & d =[1 6 9.25]. So, $W(s) = \dfrac{s+2}{s^2 + 6s + 9.25}$.

(g) A key difference between the canonical forms is that the observable form has $x_1 = y$. Hence, were we to compute the response to an initial condition of the form $\mathbf{x}_0 = [x_1 \ 0]^{tr}$, there would be no need for the procedure in (c). Is there a state space form (**A,B,C,**0) with the state $\mathbf{x} \overset{\Delta}{=} [x_1 \ x_2]^{tr} = [y \ \dot{y}]^{tr}$, so that we would not need to go through the procedure in (c) for any given initial condition? The answer is no. Is there one of the form (**A,B,C,0**) where $\mathbf{0} \overset{\Delta}{=} [0 \ 0]$ (i.e. for $\mathbf{u} \overset{\Delta}{=} [u_1 \ u_2]^{tr}$)? The answer is yes. To arrive at it, begin by recovering the plant O.D.E.

Solution: $(s^2 + 7s + 12)Y(s) = (s+2)U(s) \Rightarrow \ddot{y} + 7\dot{y} + 12y = \dot{u} + 2u$. Let $\mathbf{x} \overset{\Delta}{=} [x_1 \ x_2]^{tr} = [y \ \dot{y}]^{tr}$ and $\mathbf{u} \overset{\Delta}{=} [u_1 \ u_2]^{tr} = [u \ \dot{u}]^{tr}$.

Then $\dot{x}_1 = x_2$, and $\ddot{y} = -7\dot{y} - 12y + \dot{u} + 2u$ becomes $\dot{x}_2 = -7x_2 - 12x_1 + u_2 + 2u_1$. Hence,

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \text{and} \quad \mathbf{y} = [1 \quad 0]\mathbf{x} = \mathbf{C}\mathbf{x}.$$

(h) Use the ‘place’ command in relation to (g) to place closed loop poles at $p_{1,2} = -3 \pm 0.5i$.

Solution: Ay=[0 1 ; -12 -7]; By=[0 0 ; 2 1]; Ky=place(Ay,By,p) Ky = [-1.1 -0.4 ; -0.55 -0.2].

(i) Use the ss2tf command to recover the two CL transfer functions. Then combine them, and comment.
Solution: AAy=Ay-By*Ky; Cy=[1 0]; Dy==[0 0];

[n1 d1]=ss2tf(AAy,B,Cy,1) n1 =[1 7 17.25] d1 =[1 6 9.25]. So $W_1(s) = \frac{2}{s^2 + 6s + 9.25} = \frac{Y(s)}{U_1(s)}$

[n2 d2]=ss2tf(AAy,By,Cy,2) n2 =[0 0 1] d2 =[1 6 9.25]. So $W_2(s) = \frac{1}{s^2 + 6s + 9.25} = \frac{Y(s)}{U_2(s)}$.

Combining these gives:

$Y(s) = W_1(s)U_1(s) + W_2(s)U_2(s) = [W_1(s) + W_2(s)s]U(s)$. So $W(s) = \frac{s+2}{s^2 + 6s + 9.25} = \frac{Y(s)}{U(s)}$.

Comment We obtain the correct CL transfer function. However, using the state space form requires that we specify both the input and its derivative. For example, to compute the CL response to a unit step would require specifying $\mathbf{u} = [u\ \dot{u}]^{tr} = [1(t) \quad \delta(t)]$. This is generally not good practice, since in fact, there is only one input. Moreover, it can get nasty. Consider $\mathbf{u} = [\delta(t) \quad \dot{\delta}(t)]$. What exactly is the derivative of $\delta(t)$? Well, you might say it’s ‘complicated’.

The case of a scalar reference input is addressed in 7.5.1 of the book (p.496). It involves the concept of a *reference input*. The goal is to ensure that the CL command system has unity static gain. We might cover this at a future date. □