Lecture 15 Block Diagrams of State Space Representations (2/26/20)

***Example* 1** [(7.9) on p.469 & EXAMPLE 7.11 on p.481.]

Let $G(s)=\dfrac{s+2}{s^2+7s+12}$ with the sequence of block diagram representations in Figure 1.

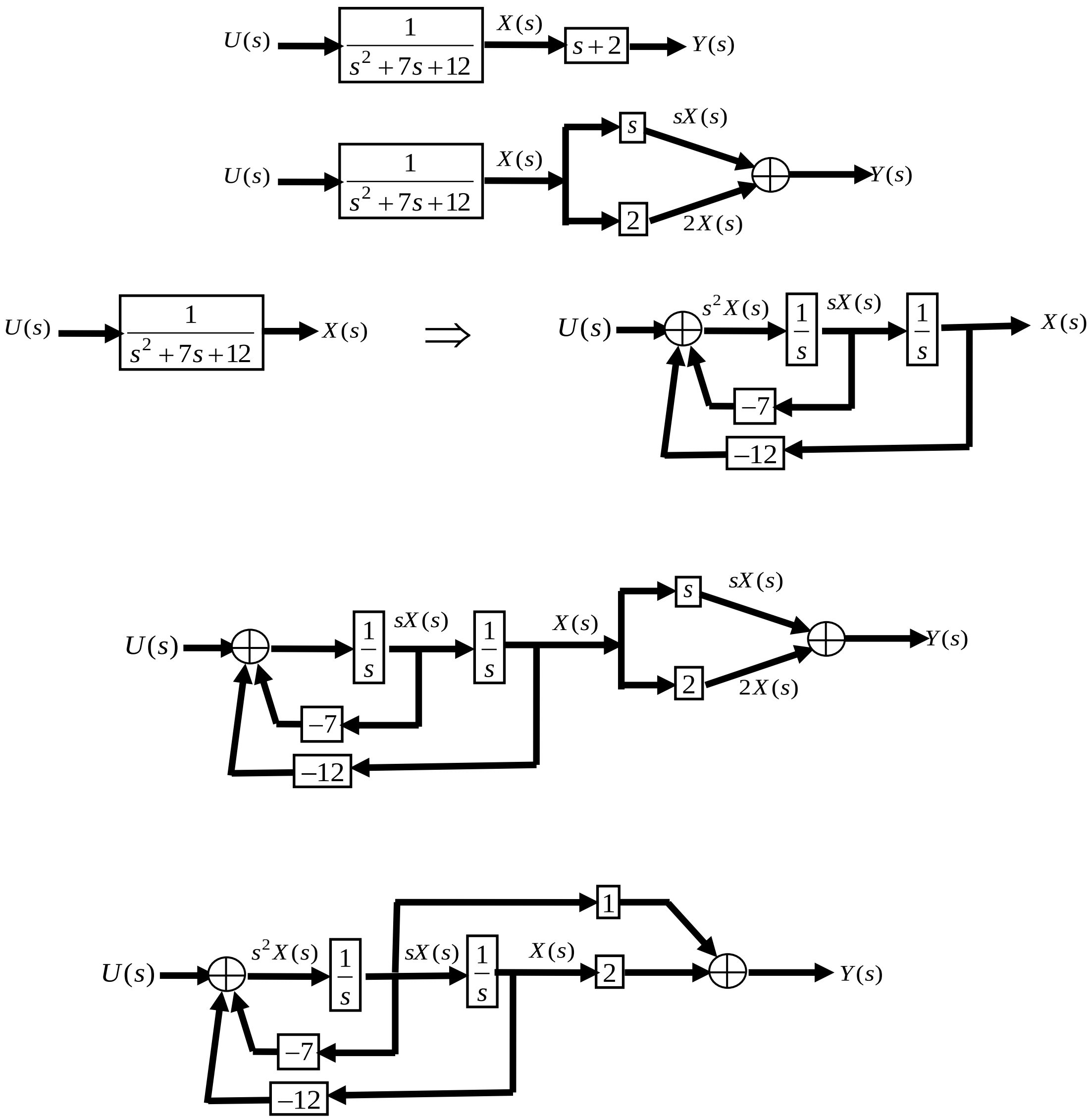


Figure 1. Sequence of block diagrams to arrive at the block diagram of the *controller canonical form.*

From the final block diagram in Figure 1 we have the following equations:

$$s^2X(s)=-7sX(s)-12X(s)+U(s) \quad \text{and} \quad Y(s)=sX(s)+2X(s).$$

Let $X_1(s)\overset{\Delta}{=}sX(s)$ and $X_2(s)\overset{\Delta}{=}X(s)$. Then these equations become:

$$sX_1(s)=-7X_1(s)-12X_2(s)+U(s) \quad \text{and} \quad Y(s)=X_1(s)+2X_2(s).$$

Remark 1. The bottom diagram in Figure 1, while more complicated than $G(s)$, provides a number of advantages:

(A1) Clear direction of actually implementing it in hardware that includes integrators, multipliers, and summing junctions. Op-amps can perform all three functions. The diagram shows that the transfer function denominator terms are fed back, while the numerator terms are fed forward.

(A2) Amenability to multi-input/multi-output (MIMO) high-dimensional settings associated with many real world settings. □

The equivalent time-domain equation are:

$$\dot{x}_1(t) = -7x_1(t) - 12x_2(t) + u(t) \quad \text{and} \quad y(t) = x_1(t) + 2x_2(t) \,.$$

These, in turn, lead to the *state space* equations:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad \text{and} \quad y = \begin{bmatrix} 1 & 2 \end{bmatrix}\mathbf{x} + 0u = \mathbf{C}x + \mathbf{D}u \,.$$

***Matlab*:**
> a=[1 7 12];   > b=[1 2];   > G=tf(b,a) = (s + 2) / (s^2 + 7 s + 12)

> [A,B,C,D]=tf2ss(b,a); A =[-7   -12 ; 1   0];   B =[1; 0]; C =[1 2]; D = 0.

<u>Conclusion</u>: The tf2ss command computes the ***controller canonical form*** of the state.

The *controllability matrix* is: $C = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$. The matrices (**A**,**B**) will be *controllable* if and only if C is nonsingular. The authors note that "controllability is a function of the *state* of the system, and cannot be decided from a transfer function." They then state (p.477) that "To discuss controllability more at this point would take us too far afield." I agree.

Now let's arrive at yet another representation.

$\ddot{y} + 7\dot{y} + 12y = \dot{u} + 2u$ gives: $\dot{y} = -7y + u + \int (-12y + 2u)dt$.

Let $x_1 = y$ and let $x_2 = \int (-12y + 2u)dt$. We then have: $\dot{x}_1 = -7x_1 + x_2 + u$. (1)

Now, $x_2 = \int (-12y + 2u)dt$ gives: $\dot{x}_2 = -12x_1 + 2u$. (2)

From (1) and (2): $\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u = \mathbf{A}_0\mathbf{x} + \mathbf{B}_0 u$ and $y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} + 0u = \mathbf{C}_0 x + \mathbf{D}_0 u$.

This is called the ***observer canonical form***. The block diagram is Figure 7.10 on p.476.

In this case the controllability matrix is: $C_0 = \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_0\mathbf{B} \end{bmatrix}$. This matrix will be singular if the transfer function zero at -2 were to drift to either -3 or -4. This is because $G(s)$ would then experience a pole/zero cancellation.

Generalization of the controller canonical state space representation:

For the single-input/single-output (SISO) transfer function

$$G(s)=\frac{b_m s^m+\cdots+b_1 s+b_0}{s^n+\cdots+a_1 s+a_0} \quad ; \quad m\le n\,,$$

the *controller* state space representation for this single-input/single-output system is:

$$\dot{\mathbf{x}}=\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}-a_{n-1} & -a_{n-2} & \cdots & -a_0\\ 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ 0 & \cdots & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}+\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix}u=\mathbf{A}\mathbf{x}+\mathbf{B}u \quad \text{and} \quad y=\begin{bmatrix}b_m & b_{m-1} & \cdots & b_0\end{bmatrix}\mathbf{x}+0u=\mathbf{C}x+\mathbf{D}u\,.$$

[Note: These equations are (7.13) on p.470 in the book.]

Remark 2. There are *many* other state space representations for a given system. The book covers them in detail. In this course we will not pursue them in such detail. Nonetheless, it is important to be aware that a given system can be represented in different ways.

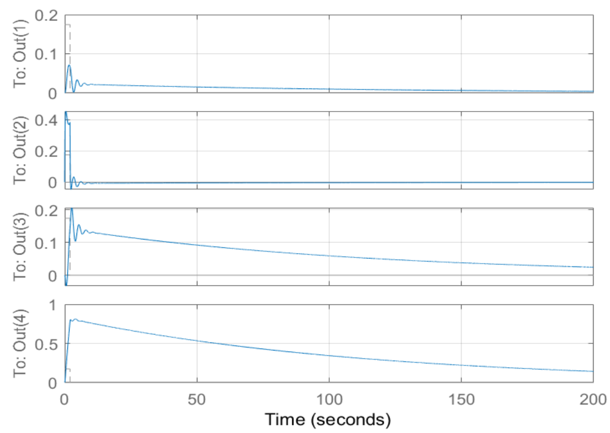
Now, if we have time, let's have some fun ☺

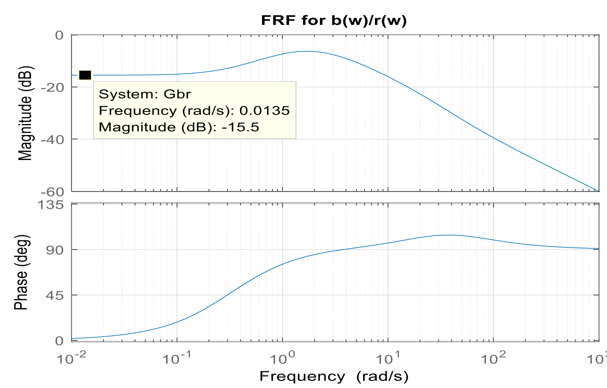
Appendix Matlab Code

```
%PROGRAM NAME: lec15.m
%EXAMPLE 1
% Lateral Perturbation Dynamics:
%x = [ b,p,r,phi]=[sideslip,rollrate,yayrate,rollangle]
A=[-.25 0 -1 .18; -16.02 -8.4 2.19 0; 4.49 -.35 -.76 0; 0 1 0 0];
B=[.07 0;-29.01 23.16; 4.55 .22; 0 0];
C = eye(4); D=zeros(4,2);
sys=ss(A,B,C,D);

T=0:.01:200; nT=length(T);
T1=0:.01:2; nT1=length(T1); %Pulse width = 2 sec.
U1 = zeros(nT,1);
U2 =[(10*pi/180)*ones(nT1,1);zeros(nT-nT1,1)];
U=[U1 , U2];
lsim(sys,U,T)
grid
%(c)
[N2,D]=ss2tf(A,B,C,D,2)
%(d):
Gbr=tf(N(1,3:5),N(3,2:5)) %This is the FUN part!

figure(30)
bode(Gbr)
title('FRF for b(w)/r(w)')
grid
```





It can be shown that the -3dB BW for a pulse of width Δ is: $\omega_{BW}\cong 2.78/\Delta=2.78/2=1.39\,rad/s$.

We also have: >> eigs(A) =[-8.4322 -0.4845 +/- 2.3329i -0.0088]. Hence, the Dutch roll damped natural frequency is 2.33 rad/sec. This is above the input -3dB BW frequency 1.39 rad/sec. Hence, this mode will not be as strongly excited as it would be, were the rudder pulse to have a shorter but greater shape.

After that response in $\beta(t)$ has decayed, what remains is very low frequency in nature (plot 1). It's early amplitude is ~0.02. The amplitude of $r(t)$ in this region is ~0.12. The ratio 0.02/0.12=1/6 = -15.6 dB; which is what is predicted in the Bode FRF plot. □