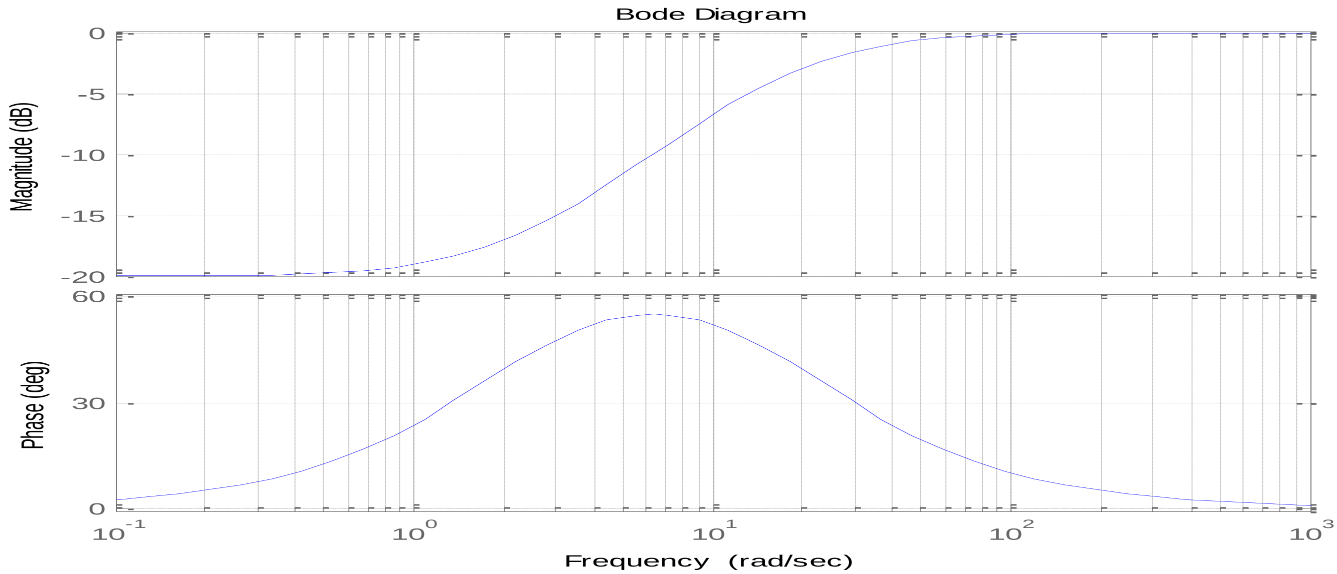
Lecture 11 **Example Problems from Chapter 6 and More**

EXAMPLE 1. [Book 6.16(a)] For a closed loop system with open loop $G(s) = \dfrac{K(s+2)}{s+20}$, draw the Bode plot of this open loop for K = 1, and then use the plot to determine the range of K for which the closed loop is stable.

Solution:



Stability: Since the open loop phase never comes close to -180^o the closed loop system will be stable for all *K*.

QUESTION: How can this be inferred from the OL Bode plot? I thought that's what the root locus was for.

REPLY QUESTION 1: What is the root locus angle criterion?

REPLY QUESTION 2: Now that you got that, what does it say for a closed loop root candidate: $s = i\omega$?

[Take a few minutes. I can wait ☺]

ANY VOLUNTEERS TO ANSWER AT THE BOARD???

EXAMPLE 2. [Book 6.17(b)] For a closed loop system with **open loop** $G(s) = \frac{K}{(s+10)(s+1)^2}$, (i) draw the Bode plot of this open loop for K = 1, (ii) draw straight-line approximations of the magnitude and phase, and (iii) then use the plot to determine the range of K for which the **closed loop** is stable.

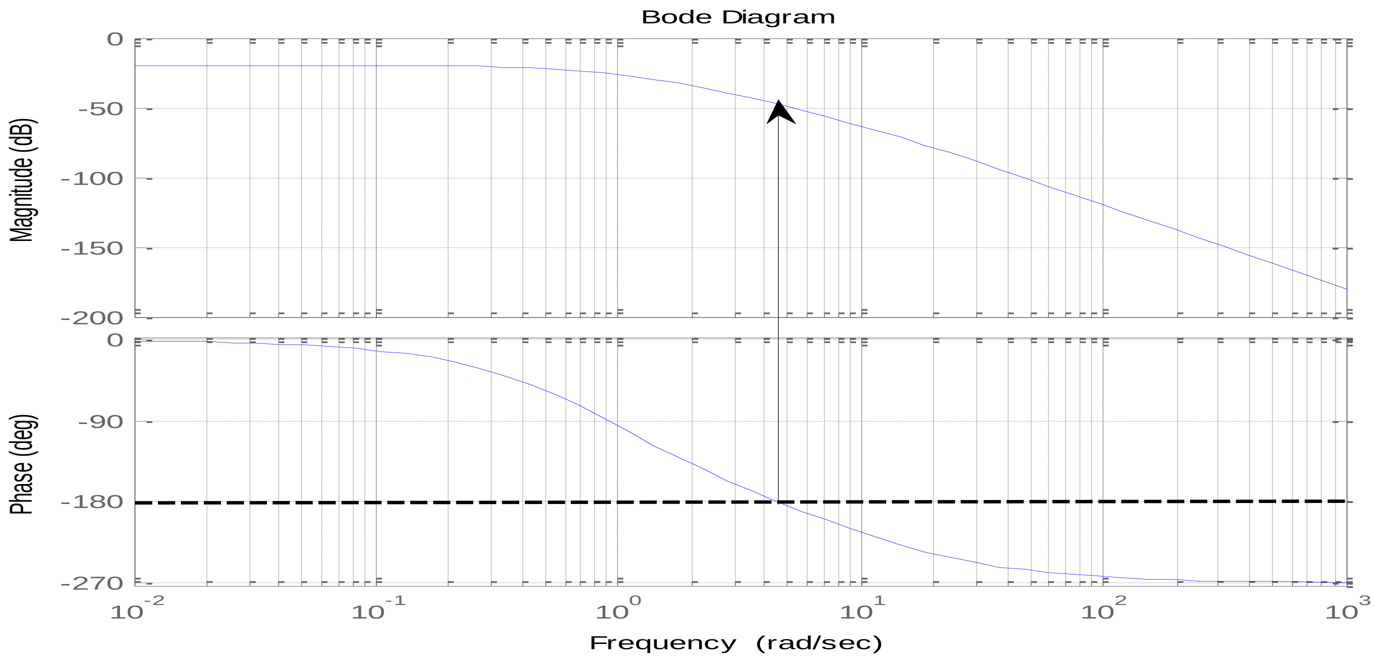
Solution:

(i)Let's use Matlab to construct the Bode plot.

(ii)Let's then insert some straight lines on it.

(iii)Any takers?

Open Loop Bode Plot:

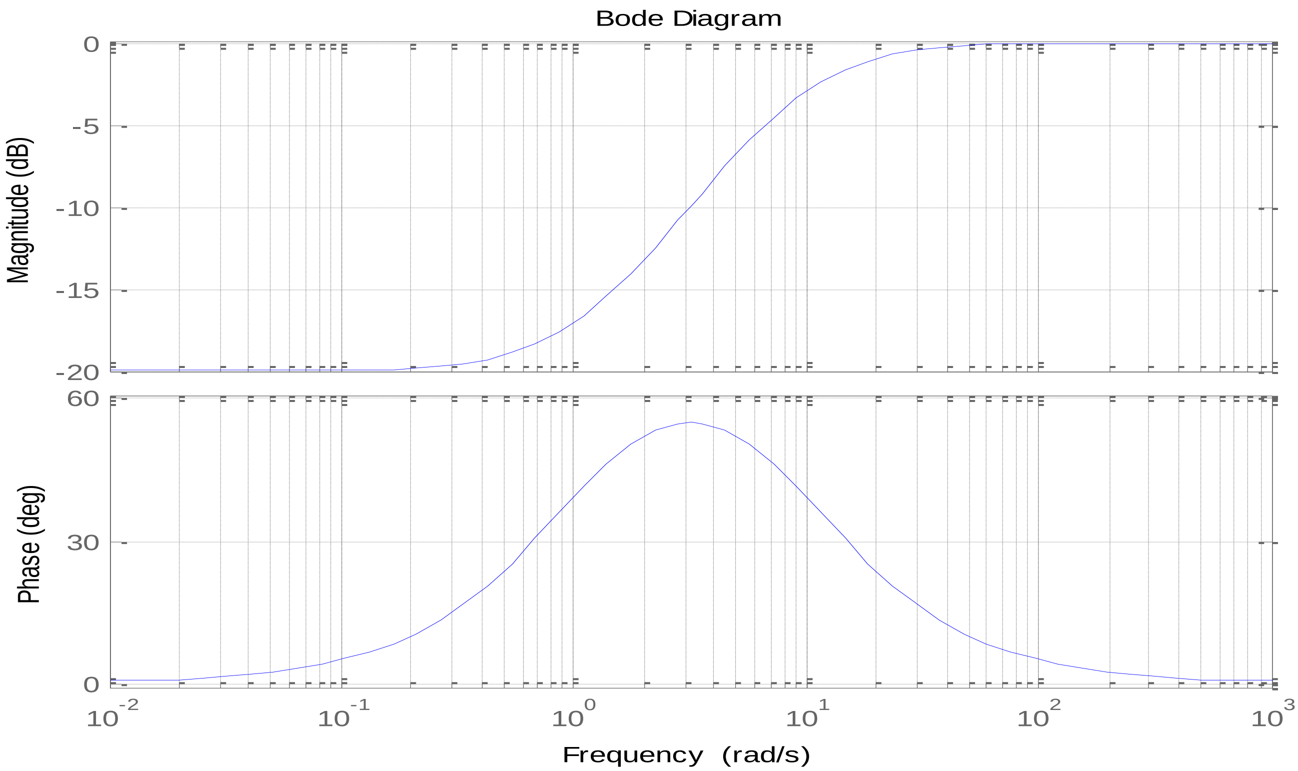


Closed Loop Stability:

At the frequency where the phase is -180°, the magnitude is ~ -47 dB. Hence, we can gain the open loop up by the ***gain margin*** K_{dB}= = 47 until the closed loop becomes unstable. The gain corresponds to $K=10^{2.35}=224$.

EXAMPLE 3. For $G_c(s) = K\left(\dfrac{s+1}{s+10}\right)$ in this example we will investigate the low and high frequency properties of the ***magnitude*** $|G_c(s)| = M(\omega)$.

(a) Give the Bode plot for ***K=1***, as well as a straight-line approximation.



(b) Describe the magnitude and phase behavior of $G_c(i\omega)$ for $\omega < 10^{-2}\, rad/\sec$:

Answer: $M(\omega) \cong -20\, dB$ and $\phi(\omega) \cong 0^o$

(c) Describe the magnitude and phase behavior of $G_c(i\omega)$ for $\omega > 10^{3}\, rad/\sec$:

Answer: $M(\omega) \cong 0\, dB$ and $\phi(\omega) \cong 0^o$

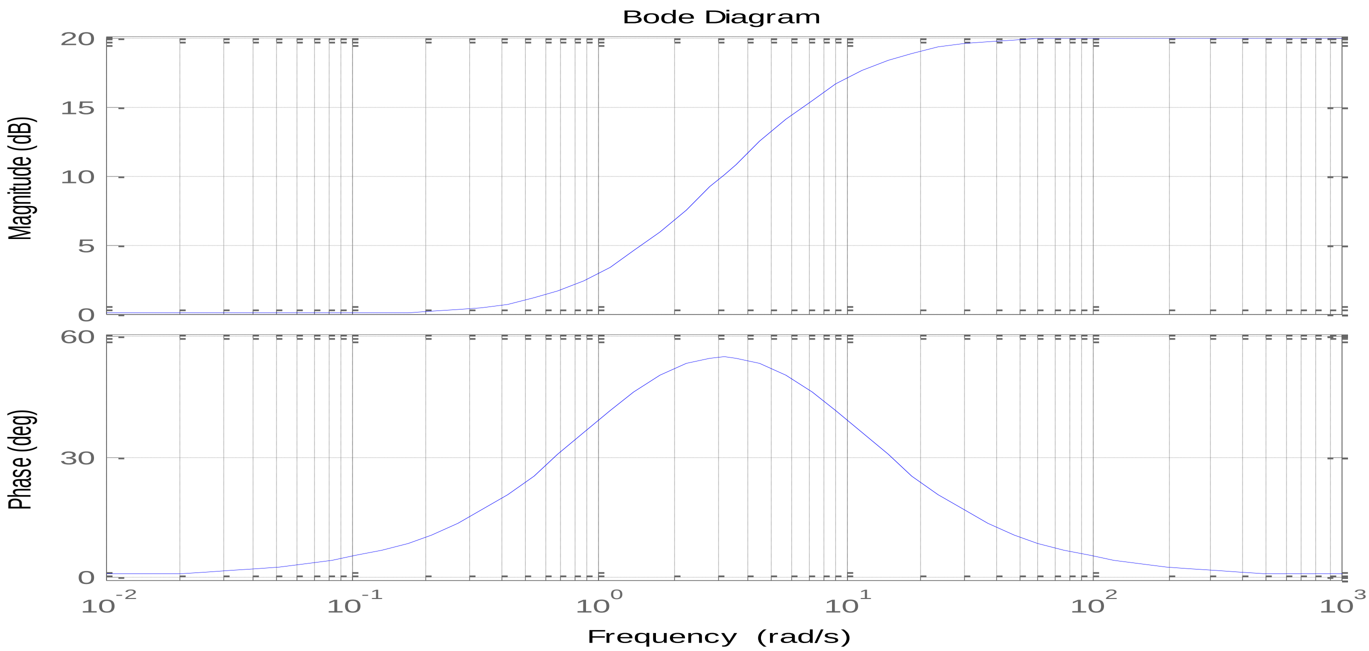
(d) Based on your answers to (b) and (c), is $G_c(s) = \dfrac{s+\omega_1}{s+\omega_2}$ a ***low pass filter*** or a ***high pass filter***?

Answer: Because it changes neither the magnitude nor the phase of sinusoid inputs at high frequencies, while it attenuates low frequency sinusoids, it is a high pass filter. [i.e. It lets high frequency inputs pass through it undistorted, while not allowing low frequency inputs to pass as much.)

(e) Find the value of *K* so that $G_c(i\omega)$ passes low frequency inputs without any distortion.

Answer: $K_{dB} = 20\, dB$ will raise $M(\omega)$ by 20 dB, so that $M(\omega) \cong 0\, dB$ for low frequencies. Hence, $K = 10$.

(f) Give the Bode plot of $G_c(s) = 10\left(\dfrac{s+1}{s+10}\right)$.



(g) Describe the magnitude and phase behavior of $G_c(i\,\omega)$ **for** $\omega > 10^3\,rad/\sec$**:**

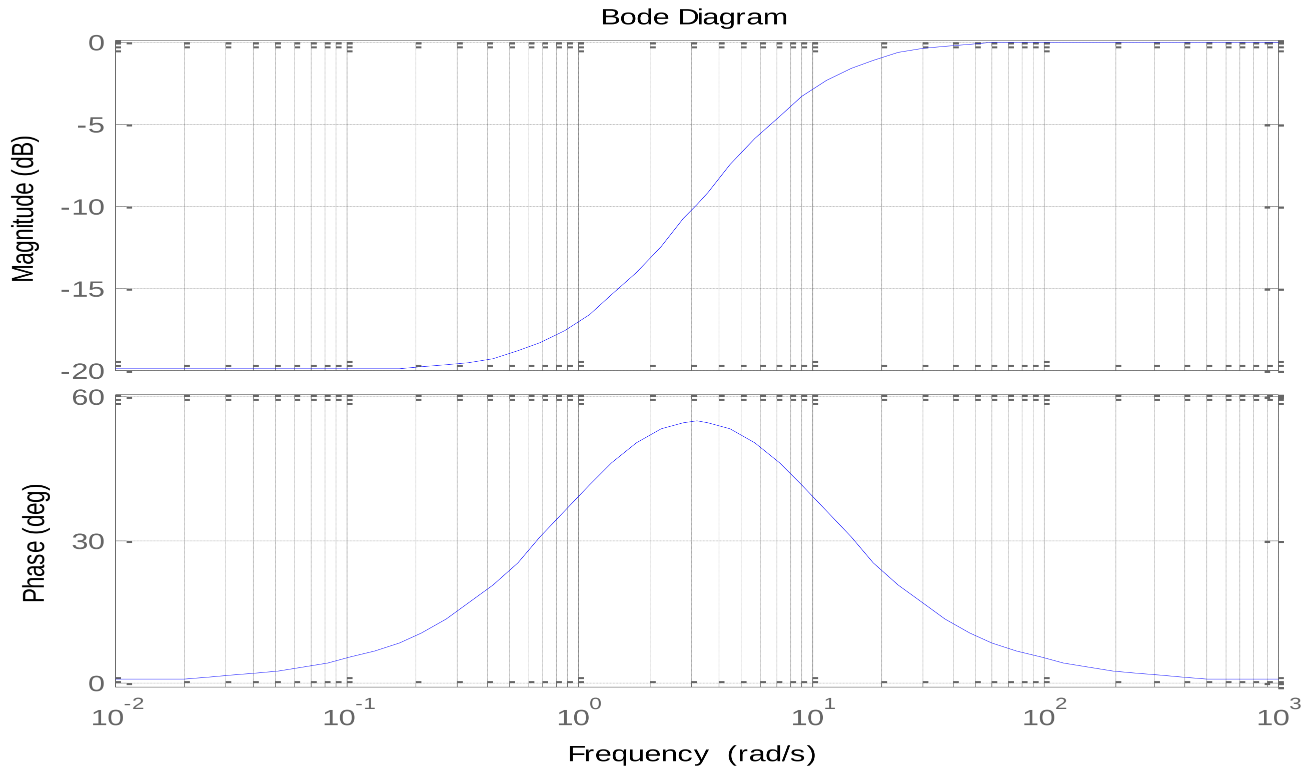
Answer: $M(\omega) \cong 20\,dB$ and $\phi(\omega) \cong 0^o$.

(h) In words, how would you describe $G_c(i\,\omega)$ **for** $\omega > 10^3\,rad/\sec$ **?**

Answer: I would describe it as a ***high frequency amplifier.***

EXAMPLE 4. For $G_c(s) = K\left(\frac{s+1}{s+10}\right)$ in this example we will focus on its use, not as a filter or an amplifier, but rather as a ***phase compensator*** .

(a) For ***K*** **= 1 the bode plot is shown below (again!) :**



Over what range of frequencies is $\phi(\omega) \ge \sim 5^o$ **?**

Answer: We can estimate this range directly from the Bode plot: $\sim [10^{-1}, 10^{1}]\, rad/\sec$.

What effect does this compensator have on the magnitude and phase of $G_p(s)$?

Answer: $G_c(i\omega)G_p(i\omega) = |G_c(i\omega)|e^{i\theta_c(\omega)}|G_p(i\omega)|e^{i\theta_p(\omega)} = \{|G_c(i\omega)||G_p(i\omega)|\}e^{i\{\theta_c(\omega)+\theta_p(\omega)\}}$.

Hence, in dB, the effect is: $20\log\{|G_c(i\omega)||G_p(i\omega)|\} = 20\log|G_c(i\omega)| + 20\log|G_p(i\omega)| = M_c(\omega)_{dB} + M_p(\omega)_{dB}$.

In relation to phase: $\theta_c(\omega) + \theta_p(\omega)$.

In summary, the effect is to add (or subtract) dB, and to add (or subtract) phase.

Derivation of Phase Compensator Design Equations.

Here we will derive the mathematical formulas needed to design a unity static gain ***phase-lead*** compensator:

$$G_c(s) = \left(\frac{\omega_2}{\omega_1}\right)\left(\frac{s+\omega_1}{s+\omega_2}\right) \text{ with } \omega_1 < \omega_2 \,. \tag{1}$$

First, we write the compensator FRF: $G_c(i\omega) = \left(\frac{\omega_2}{\omega_1}\right)\left(\frac{\omega_1 + i\omega}{\omega_2 + i\omega}\right)$.

Hence: $M(\omega) \overset{\Delta}{=} |G_c(i\omega)| = \left(\frac{\omega_2}{\omega_1}\right)\frac{\sqrt{\omega_1^2+\omega^2}}{\sqrt{\omega_2^2+\omega^2}}$ and $\theta(\omega) = \tan^{-1}(\omega/\omega_1) - \tan^{-1}(\omega/\omega_2)$.

(a) Find the frequency at which $\theta(\omega)$ is maximum.

Solution:

Now, before I go any further, let me explain why I am carrying out the calculation in detail.. It is for those students who have the interest in knowing how formulas are obtained, and who value applying their mathematical skills once in a while; especially to problems in an area of special interest to them. And so: If you want to sleep through this derivation, have sweet dreams!. It will not be on any homework or exam.

OK. Here's the first of the two 'tricks' that I use to solve this problem: "Recall" (ha! ha!) the trig. identity:

$$\tan(\theta_1 - \theta_2) = \frac{\tan(\theta_1) - \tan(\theta_2)}{1 + \tan(\theta_1)\tan(\theta_2)} \,.$$

To use this, let $\theta_1 = \tan^{-1}(\omega/\omega_1)$ and $\theta_2 = \tan^{-1}(\omega/\omega_2)$. We then have

$$\tan[\phi(\omega)] = \frac{(\omega/\omega_1) - (\omega/\omega_2)}{1 + (\omega/\omega_1)(\omega/\omega_2)} = \frac{\omega\omega_2 - \omega\omega_1}{\omega_1\omega_2 + \omega^2} \quad , \text{ and hence:}$$

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega\omega_2 - \omega\omega_1}{\omega_1\omega_2 + \omega^2}\right)$$

To find the frequency $\omega_{\max}$ where $\phi(\omega)$ achieves its maximum value, we need to solve $\frac{d\phi(\omega)}{d\omega} = 0$. Now, on the surface it may seem like this is a real pain, due to the arctan functions. *However…* recall that the arctan is a monotonically increasing function on the interval $(-\pi/2, \pi/2)$. To see why this matters, recall the chain rule for differentiation (the second trick):

$$\frac{dg[f(\omega)]}{d\omega} = \frac{dg}{df} \bullet \frac{df}{d\omega} \,.$$

If $g(x)$ is strictly monotonic, that means that it derivative is strictly greater than (or less than) zero. Hence, if in this case we set the above equation to zero, then it must be that $\frac{df}{d\omega} = 0$. Applying this second 'trick', we get

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$$\frac{d\phi(\omega)}{d\omega} = 0 = \frac{d}{d\omega}\left(\frac{\omega\omega_2 \quad - \quad \omega\omega_1}{\omega_1\omega_2 \quad + \quad \omega^2}\right) = \frac{(\omega_2 - \omega_1)(\omega_1\omega_2 + \omega^2) - \omega(\omega_2 - \omega_1)2\omega}{(\omega_1\omega_2 \quad + \quad \omega^2)^2}.$$

For the rightmost quantity to equal zero, its numerator must equal zero; that is:

$$0 = (\omega_2 - \omega_1)(\omega_1\omega_2 + \omega^2) - \omega(\omega_2 - \omega_1)2\omega = (\omega_2 - \omega_1)(\omega_1\omega_2 + \omega^2 - 2\omega^2) = (\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2).$$

The solution to this is $\omega_{\max} = \sqrt{\omega_1\omega_2}$ = the frequency where $\phi(\omega)$ achieves its maximum.

(b) Find the expression for the controller *magnitude* at $\omega_{\max} = \sqrt{\omega_1\omega_2}$.

Solution: This is 'relatively' easy:

$$|M(\omega_{\max})| = \left(\frac{\omega_2}{\omega_1}\right)\sqrt{\frac{(\omega_1^2 + \omega_{\max}^2)}{(\omega_2^2 + \omega_{\max}^2)}} = \sqrt{\frac{\omega_1^2 + \omega_1\omega_2}{\omega_2^2 + \omega_1\omega_2}} = \left(\frac{\omega_2}{\omega_1}\right)\sqrt{\frac{\omega_1(\omega_1 + \omega_1)}{\omega_2(\omega_1 + \omega_1)}}.$$

Hence, $|M(\omega_{\max})| = \sqrt{\frac{\omega_2}{\omega_1}}$. Or: $M(\omega_{\max})_{dB} = 10\log\left(\frac{\omega_2}{\omega_1}\right)$

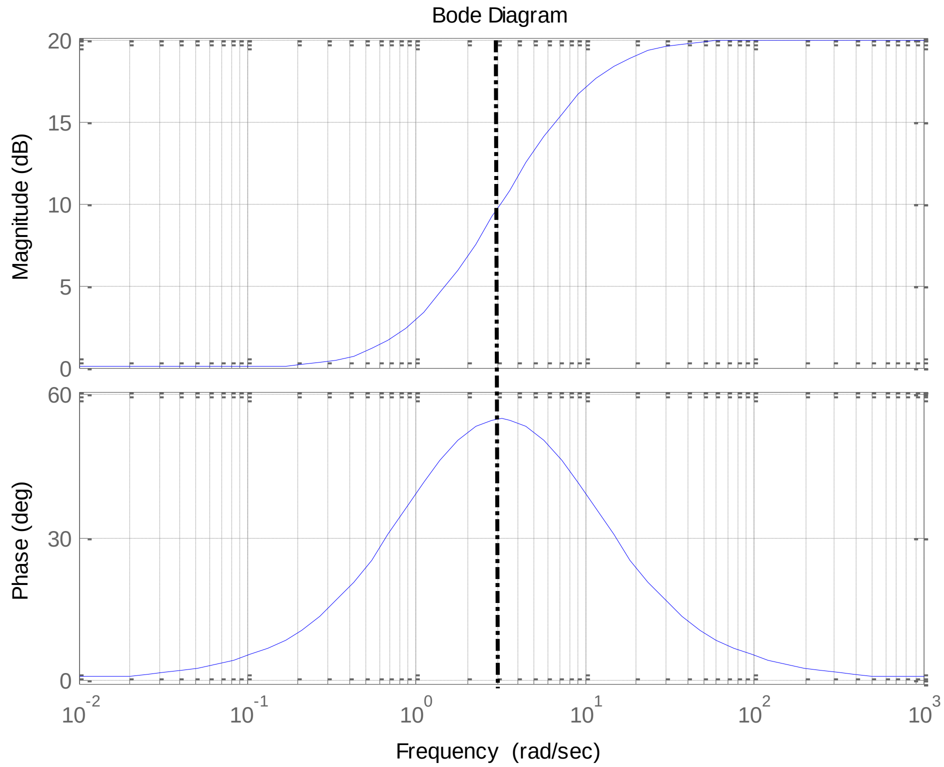
NOTE: Even though we will often use this as a phase compensator, it is crucial to not that, along with contributing $\phi(\omega_{\max}) = \phi_{\max}$, it also contributes $M(\omega_{\max})_{dB} = 10\log\left(\frac{\omega_2}{\omega_1}\right)$ at $\omega_{\max} = \sqrt{\omega_1\omega_2}$.

Now, the static gain of the controller is $|G_c(\omega << \omega_1)| = 1$ and its high frequency gain is $|G_c(\omega >> \omega_2)| = \frac{\omega_2}{\omega_1}$. If we convert these gains to decibels, then we have:

$$|G_c(\omega << \omega_1)|_{dB} = 0\,dB \quad \text{and} \quad |G_c(\omega >> \omega_2)|_{dB} = 20\log\left(\frac{\omega_2}{\omega_1}\right).$$

But we also have $|G_c(i\omega_{\max})|_{dB} = 10\log\left(\frac{\omega_2}{\omega_1}\right)$. In words, the controller achieves half of its total dB gain $\omega_{\max} = \sqrt{\omega_1\omega_2}$.

To visualize this, let $\omega_1 = 1$ and $\omega_1 = 10$. The Bode plot for this controller is shown below.



We see that $\omega_{\max} = \sqrt{\omega_1\omega_2} = \sqrt{10}$ is where the phase is maximum, and where the gain is half way from 0 dB to 20 dB; that is, 10 dB.

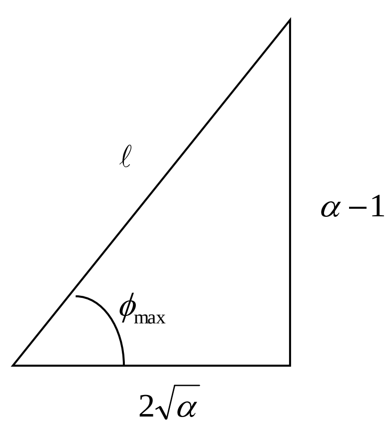
(c) Derive the expression for $\phi(\omega_{\max})$.

Solution: To this end, define $\omega_2 / \omega_1 \stackrel{\Delta}{=} \alpha$. [**NOTE:** On p.349 the term α used by the authors is $\omega_1 / \omega_2 \stackrel{\Delta}{=} \alpha$! Hence, my α is the ***inverse*** of theirs.]

We then have

$\phi(\omega_{\max}) = \tan^{-1}(\sqrt{\alpha}) - \tan^{-1}(1/\sqrt{\alpha})$. Applying the above trig. identity gives

$\tan(\phi_{\max}) = \dfrac{\alpha - 1}{2\sqrt{\alpha}}$. Now consider the following right triangle:



We see that $\ell = \sqrt{4\alpha + (\alpha - 1)^2} = 1 + \alpha$. Hence

$\sin(\phi_{\max}) = \dfrac{\alpha - 1}{\alpha + 1}$, or $\boxed{\alpha = \dfrac{1 + \sin(\phi_{\max})}{1 - \sin(\phi_{\max})}}$

We can now summarize the design procedure for using a lead controller to increase the open loop phase by a specified amount $\phi_{\max}$:

1. Having specified $\phi_{\max}$, compute $\alpha = \dfrac{1+\sin(\phi_{\max})}{1-\sin(\phi_{\max})}$

2. Since $\omega_2 / \omega_1 \stackrel{\Delta}{=} \alpha$, by knowing α we know the ratio of the controller break frequencies.

3. Decide at what frequency $\omega_{\max}$ this phase gain $\phi_{\max}$ should occur. This decision will often be made in conjunction with other closed loop specifications (e.g. error constants, etc.), and so no one procedure can be described here.

Implementation of the Design Equations- For *unity static gain* $G_c(s) = \left(\dfrac{\omega_2}{\omega_1}\right)\left(\dfrac{s+\omega_1}{s+\omega_2}\right) = \alpha\left(\dfrac{s+\omega_1}{s+\omega_1\alpha}\right)$:

- ***When $G_c(s)$ is to modify open loop phase in order to achive a closed loop PM***

1. Compute the current *PM*, and then compute the needed $\phi_{\max}$ (+ a ‘cushion’).

2. Compute an initial value for $\alpha = \dfrac{1+\sin(\phi_{\max})}{1-\sin(\phi_{\max})}$.

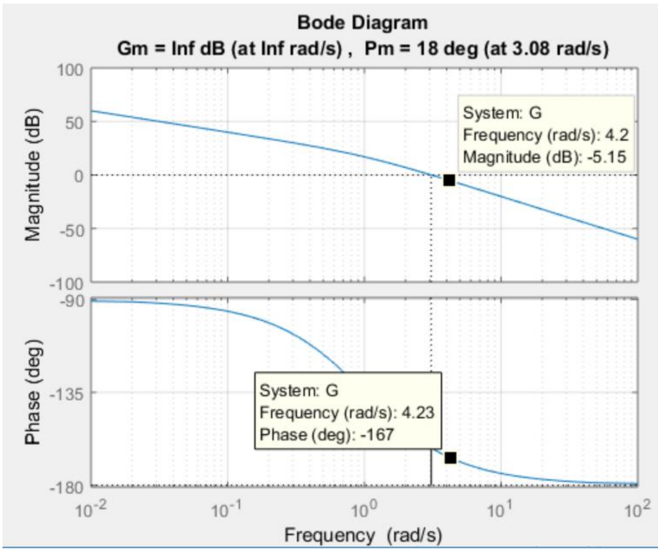
3. Locate the frequency, ω_0 , where the current open loop has $|G_{O.L.}(i\omega_0)|_{dB} = -\alpha_{dB}/2$. [This is because, if you were to now re-center the compensator at ω_0 , this would become $\omega_{gc}^{(new)}$.]

4. Compute the open loop phase $\theta_{O.L}^{(new)}(\omega_0) = \theta_{O.L}^{(old)}(\omega_0) + \phi_{\max}$.

5. Compute $180^o + \theta_{O.L}^{(new)}(\omega_0)$. If this value is close to the required *PM, ‘mission accomplished’*. If this value is deemed to be unacceptably low, then return to 1. and add a larger ‘cushion’. [If this value is deemed to be unacceptably large (which rarely happens!), then return to 1. and reduce the ‘cushion’.]

6. Compute $\omega_1 = \omega_0 / \sqrt{\alpha}$, followed by $\omega_2 = \omega_1\alpha$

Example 5. [See book Example 6.15 on p. 367] Consider a unity feedback system with open loop transfer function $G(s) = 10/s(s+1)$ with FRF at right. Suppose that we desire $PM = 45^o$.



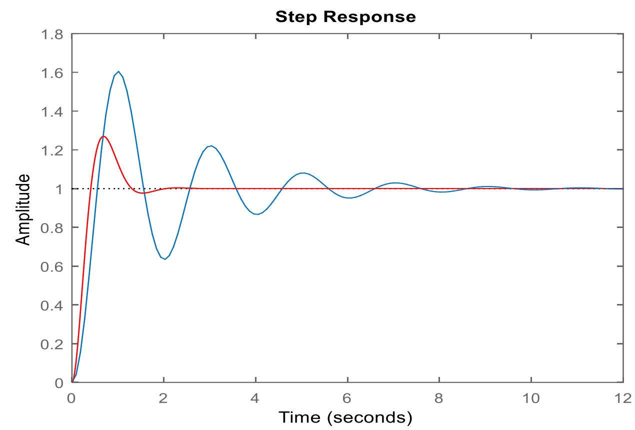
1. Using margin(G) we get $PM = 18^o$. We will choose $\phi_{\max} = 45^o - 18^o + 5^o = 32^o$ (i.e. a 5^o cushion).

2. Then a=(1+sind(32))/(1-sind(32)) = 3.2546, or adB=20*log10(a) =10.25 dB.

3. For $|G_{O.L.}(i\omega_0)|_{dB} = -\alpha_{dB}/2 = -5.125dB$ sing the data cursor, we find $\omega_0 \cong 4.2r/s$.

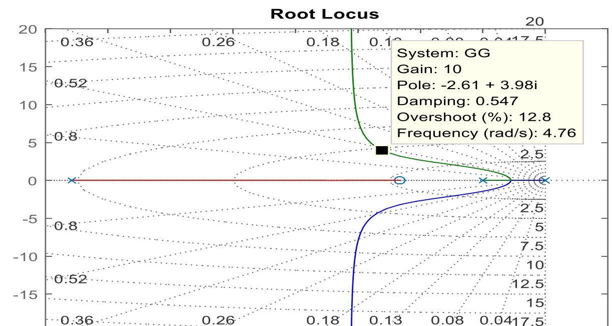
4. Using the data cursor, we then find $\theta_{O.L}^{(old)}(\omega_0) = -167^o$, hence $\theta_{O.L}^{(new)}(\omega_0) = -167^o + 32^o = -135^o$.

5. $180^o - \theta_{O.L}^{(new)}(\omega_0) = 180^o - 135^o = 45^o \equiv PM$. *Mission accomplished* (in a single try!)



6. $\omega_1 = \omega_0 / \sqrt{\alpha} = 2.33 r/s$, followed by $\omega_2 = \omega_1 \alpha = 7.58 r/s$

Hence, $G_c(s) = \alpha\left(\frac{s+\omega_1}{s+\omega_2}\right) = 3.25\left(\frac{s+2.33}{s+7.58}\right)$. The closed loop step responses for the controllers $G_{C_1}(s) = 10$ and $G_{C_2}(s) = 10 \times 3.25\left(\frac{s+2.33}{s+7.58}\right)$ are shown at right. Clearly, the inclusion of the *PM* specification resulted in a huge improvement.



Before leaving this example, in order to better tie this design method to the root locus pole placement method, we offer the root locus corresponding to the open loop system

$$G(s) = K \times 3.25\left(\frac{s+2.33}{s+7.58}\right)\left(\frac{1}{s(s+1)}\right).$$

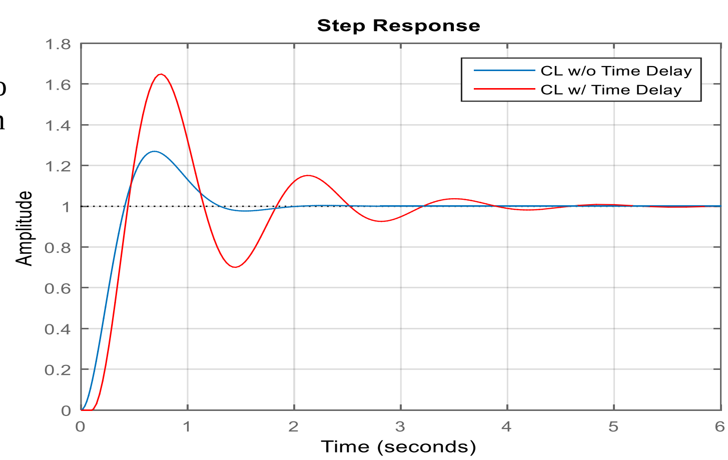
The root locus plot at the right reveals that for $K = 10$ (required to satisfy the steady state error specification), the complex closed loop poles are *maximally damped.* [As an aside: Whereas the data cursor states that those poles correspond to 12.% overshoot, we see from the above step response, that the actual overshoot is ~20%. This increase is due to the closed loop zero that the phase compensator introduced.]

QUESTION: In view of the above, why didn't we simply design the controller via the root locus pole-placement method? It seems like we are just learning a new method that gives the same results we would have gotten using a method we already know?

ANSWER: Recall that the frequency domain method never actually used direct knowledge of the plant transfer function. Even though we obtained the plant FRF by assuming it was known, very often a plant FRF is obtained from experimental data. Hence, the answer is: Because often we do not have a mathematical model for the plant transfer function, the frequency domain method can be more appropriate.

As reasonable as the above answer is, there is yet another additional answer that is often as, if not more important. To this end, suppose that there is a 0.1 sec. time delay between when the error is computed and when it is received by the controller. To incorporate this delay into the open loop, recall that for $e(t) \leftrightarrow E(s)$, then $e(t-t_0) \leftrightarrow e^{-st_0}E(s)$. And so now the final open loop transfer function is:

$$G(s) = 10 \times 3.25\left(\frac{s+2.33}{s+7.58}\right)\left(\frac{1}{s(s+1)}\right) \times e^{-0.1s} .$$



The closed loop unit step responses without and with this time delay included are shown at right. Clearly, the time delay had a destabilizing effect. The term e^{-st_0} in the open loop transfer function means that it is no longer a ratio of polynomials, from which one simply obtains poles and zeros. Hence, now, the root locus method *does not even apply*. Even so, this term is trivial to incorporate into the open loop FRF. This is because $e^{-i\omega t_0}$ has a magnitude of 1.0 (and so does nothing to the open loop magnitude, and contribute angle $\theta_{t_0}(\omega) = -\omega t_0 = -0.1\omega$. And so, at the new gain crossover frequency $\omega_0 \cong 4.2\, r/s$ this time delay will reduce the open loop phase by $\theta_{t_0}(\omega) = 0.1(4.2)(180/\pi) \cong 24^o$. Hence, when the delay is included, the closed loop $PM = 45^o - 24^o = 21^o$. OUCH! Basically, we are back to where we were prior to incorporating the phase compensator. Yes, that IS true. Nowever, had that compensator NOT been incorporated, the closed loop PW would be $PM = 18^o - 24^o = -6^o$. In layman's terms, we would have been SOL.

- ***When $G_c(s)$ is to modify open loop magnitude*:**