Lecture 10 The System Frequency Response Function

Even though humanoids have a preference for viewing the world in the time domain in their personal lives, when it comes to their professional lives, engineers tend to view the world in the frequency domain. Before offering a psychoanalysis of this aberrant behavior, it is appropriate that we first elaborate on what the frequency domain actually is.

Consider any function of time, $x(t)$ defined over the continuous-time interval $-\infty < t < \infty$. Furthermore, assume that $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$, (i.e. $x(t)$ is square-integrable). This condition holds for many transients. We then have

Definition 1 The ***Fourier Transform*** of $x(t)$ is defined as $X(\omega) \stackrel{\Delta}{=} \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$ for $-\infty < \omega < \infty$. (1)

While $x(t)$ lives in the time domain, $X(\omega)$ lives in the frequency domain. Obviously, then next question is: Why would one care to live in the frequency domain?

The answer is that the frequency domain can provide insight into $x(t)$ that is lacking in the time domain. This is especially true in the setting where $x(t)$ is the output of a linear system having an input $f(t)$. Recall that if $f(t) = F_0 \sin(\omega t + \phi)$, then the steady state response will be $x(t) = [M(\omega)F_0]\sin[\omega t + \phi + \theta(\omega)]$, where $M(\omega)$ and $\theta(\omega)$ are the magnitude and phase, respectively, of the ***frequency response function*** (FRF) $G(i\omega)$. And so, by identifying the frequency structure of the input, we can predict the frequency structure of the steady state response. We will now proceed to go into more detail in relation to the FRF.

The Frequency Response Function (FRF)

Consider a system with impulse response $g(t)$. The system transfer function is the Laplace transform:

$$G(s) = \int_0^{\infty} g(t)e^{-st}dt \,. \tag{2}$$

The frequency response function (FRF) is then:

$$G(i\omega) = \int_0^{\infty} g(t)e^{-i\omega t}dt \quad \text{for } -\infty < \omega < \infty \,. \tag{3}$$

Comparing (3) to (1), we see that the FRF is the Fourier Transform of the impulse response $g(t)$. The reason that the lower limit of integration in (3) is zero is that we have implicitly assumed that $g(t) = 0$ for $t < 0$. Such a system is said to be a causal system; that is, a system where the output cannot precede the input. An example of a noncausal system is one where analysis can proceed both forward and backward in time (e.g. in analysis of off-line data). We have up to this point, and will continue restrict our attention to only causal systems. We will also continue to restrict attention to inputs that equal zero before time zero.

At this point it is fair to ask: What is the meaning of negative frequency, as indicated in (3)? Good question!

Recall that the Laplace transform is in relation to $s = \sigma + i\omega$. There is nothing that prevents ω from being negative. In fact, consider a second order underdamped system having poles $s_{1,2} = -\zeta\omega_n \pm i\omega_d$. One might ask the similar question: What is the meaning of a negative damped natural frequency? In this case, the answer is simple if one considers the complementary solution to the ODE: $C_1e^{s_1t}+C_2e^{s_2t}$. Since this solution is real-valued, and since $s_2=\bar{s}_1$, it must be that $C_2=\bar{C}_1$. Hence, $C_1e^{s_1t}+C_2e^{s_2t}=Ce^{s_1t}+\bar{C}e^{\bar{s}_1t}=2\,\mathrm{Re}(Ce^{s_1t})$. Hence, we see that the negative frequency is simply a mathematical necessity in order that we have a real-valued solution.

We are now in a position to explain why the FRF is plotted only for nonnegative frequencies. In (3) the impulse response $g(t)$ is real-valued. Hence, $G[i(-\omega)]=\bar{G}(i\omega)$. If we express $G(i\omega)$ in the polar form $G(i\omega)=M(\omega)e^{i\theta(\omega)}$, then we see that $G[i(-\omega)=M(\omega)e^{i[-\theta(\omega)]}$. In words, $G[i(-\omega)]$ has the same magnitude as $G(i\omega)$, and its phase is the negative of the phase of $G(i\omega)$. Hence, plotting over negative frequencies would give redundant information. However, it must be repeated that $G(i\omega)$ is defined over negative frequencies. We simply don't plot it over them.

We will now proceed to address the more practical elements of the FRF.

Example 1 Consider a system with transfer function $\dfrac{X(s)}{F(s)}=G(s)\;=\dfrac{g_s}{\tau s+1}$. The system FRF is $G(i\omega)\;=\dfrac{g_s}{1+i(\omega\tau)}$. In polar form, we have $G(i\omega)\;=M(\omega)e^{i\theta(\omega)}$, where

$M(\omega)\;=\dfrac{g_s}{\sqrt{1+(\omega\tau)^2}}$ and $\theta(\omega)\;=-\tan^{-1}(\omega\tau)$. A plot of this FRF for $g_s=1$ and $\tau=1$ is shown at right.

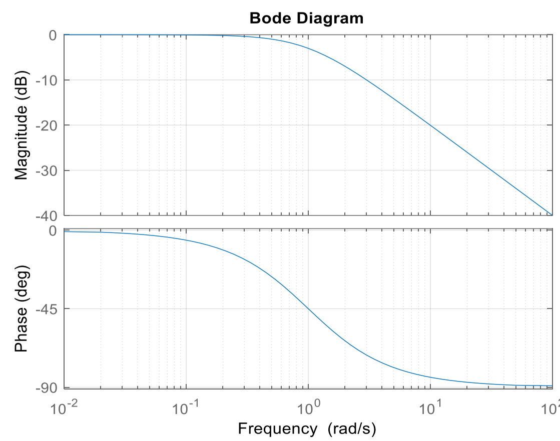

Figure 1 FRF for $G(s)=1/(s+1)$.

There are a number of points worth noting in relation to Figure 1:

1. It is labeled as a Bode plot. The Matlab command bode(G) assumes that G is a transfer function. Hence, a Bode plot is exactly the FRF of the system $G(s)$. What distinguishes it from a standard FRF plot is that the amplitude is in decibels and the frequency is plotted in logarithmic form.
2. The magnitude has units of *decibels* (dB). $M_{dB}\overset{\Delta}{=}20\log_{10}(M)$.
3. The frequency axis has log spacing.
4. One could obtain the plot by using the Matlab command 'freqs'. The 'bode' command is easier and better-suited for our needs.

***The Information in Figure* 1-**

As noted above, for a stable system with input $f(t)=F_0\sin(\omega t)$, the steady state output will be $x(t)=F_0M(\omega)\sin[\omega t+\theta(\omega)]$. And so, the system output is a sinusoid having the same frequency as the input. Its amplitude is scaled by an amount $M(\omega)$ and its phase is changed by an amount $\theta(\omega)$. For example, suppose that the input to $G(s)$ is $x(t)=2\sin(10t)$. From Figure 1 we have:

$$M_{dB}=-20=20\log_{10}(M)\Rightarrow M=0.1 \qquad \text{and} \qquad \theta(\omega)\cong-85^o.$$

So, the steady state response to this input will be $y(t) = 0.2\sin(10t - 85^o)$. To verify this, the code below was used to obtain the plot at right.

```
>> t=0:.01:10;
>> x=2*sin(10*t);
>> y=lsim(G,x,t);
>> plot(t,x)
>> hold on
>> plot(t,y,'r')
>> grid
```

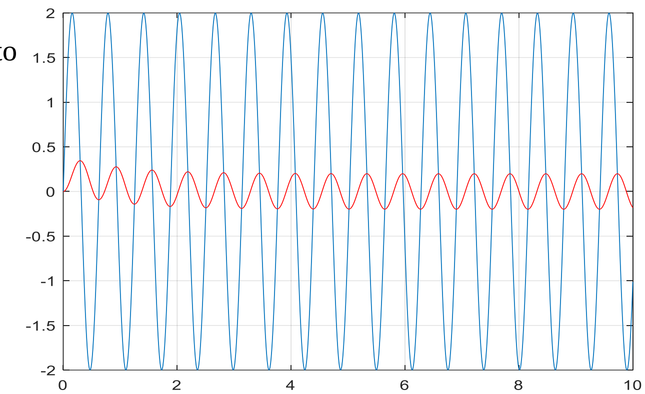

Figure 2. Total response for input $x(t) = 2\sin(10t)$.

Notice that only after the initial transient has died out, does the response become sinusoidal. □

Most students who take a course in feedback control systems will not become control systems engineers. Even so, many will encounter FRFs in their work. One area in which they occur is in relation to instrumentation (e.g. sensors).

Examples of Instrument Frequency Response Functions

A few examples of sensors include:

1.Microphones: http://blog.shure.com/how-to-read-a-microphone-frequency-response-chart/

2.Accelerometers: https://www.endevco.com/news/archivednews/2009/2009_09/TP328.pdf

3.Rate Gyros:
https://www.google.com/search?q=rate+gyro+frequency+response+function&sa=N&biw=1152&bih=529&tbm=isch&tbo=u&source=univ&ved=0ahUKEwjfup_4uqfLAhVDmYMKHRX-CiI4FBCwBAga&dpr=1.67

[For those with insomnia, and/or want to go to MIT: https://www.youtube.com/watch?v=fKaZeD70p8I **]**

In sensing pressure, velocity, sound, etc. the input to the sensor is the variable of interest. However, the sensor output is typically voltage. Hence, it is imperative that there be a 1:1 relation between the units of the variable of interest and the voltage output from the sensor. Furthermore, this relation must hold over an entire range of specified frequencies. This leads to

Definition 2 The *bandwidth* (BW) of a system is the range of frequencies in which useful information is required.

This definition is intentionally vague, since what comprises useful information is in the eye of the beholder.

Example 2 Consider the transfer function that relates forces on a wing to the wing tip displacement:

$$\frac{X(s)}{F(s)} = G(s) = \frac{50}{s^2 + 2s + 100}. \tag{3}$$

The poles of (3) are: $s_{1,2} = -1 \pm i9.95$. It is a relatively low-damped system with $\zeta = 0.1$. It has a damped natural frequency $\omega_d = 9.95\, rad/s$, or $f_d = 1.58\, Hz$. The FRF is: $G(i\omega) = \dfrac{50}{100 - \omega^2 + i2\omega}$. Hence, the FRF squared magnitude is

$$M^2(\omega) = \frac{2550}{(100-\omega^2)^2 + 4\omega^2}. \tag{4}$$

We will now find the frequency at which (4) is a maximum.

To this end, let's express $G(s)$ in the more general form:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{5}$$

Notice the (5) was chosen to have unity static gain. This is merely a convenience.

Then
$$M^2(\omega) = \frac{\omega_n^4}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2} = \frac{1}{(1-(\omega/\omega_n)^2)^2 + 4\zeta^2(\omega/\omega_n)^2} = \frac{1}{(1-r)^2 + 4\zeta^2 r} = q(r) \tag{6}$$

where we have defined $r \overset{\Delta}{=} (\omega/\omega_n)^2$. Setting the derivative of (6) equal to zero and solving for r gives: $r = 1 - 2\zeta^2$. Hence, (6) is maximum at frequency:

$$\omega_{\max} = \omega_n\sqrt{1-2\zeta^2}. \tag{7a}$$

The magnitude of the FRF at the frequency (7a) is:

$$M(\omega_{\max}) = \frac{1}{2\zeta\sqrt{1-\zeta^2}}. \tag{7b}$$

The frequency (7a) where the magnitude of the FRF is a maximum is called the ***resonance*** (or resonant) ***frequency***. In many settings, the structure resonance is low-damped (i.e. $\zeta << 1$.) In this case, (7) becomes:

$$\omega_{res} = \omega_n\sqrt{1-2\zeta^2} \cong \omega_n\sqrt{1-\zeta^2} = \omega_d \cong \omega_n. \tag{8a}$$

$$M(\omega_{res}) \cong \frac{1}{2\zeta} \overset{\Delta}{=} Q. \tag{8b}$$

Hence, in the case $\zeta << 1$, from (8a) we see that the resonance frequency is nearly the same as the damped natural frequency; which, in turn, is nearly the same as the undamped natural frequency. From (8b) we see that the magnification at resonance is inversely proportional to ζ. The magnification (8b) is often referred to as the Q factor of the resonance.

Now, returning to the example at hand, we have $\zeta = 0.1$, which is, arguably, quite small. Hence, from (8) the resonance frequency is $\omega_{res} \cong 10\, rad/s$, and $Q = 5$, or $Q_{dB} \cong 14\, dB$. The FRF associated with (3) is shown at right. The data cursor information verifies these approximations. Note that Q is the magnification relative to the static gain.

Hence, in order to minimize the wing tip vibration, we would want to avoid inputs have significant energy at the resonance frequency. □

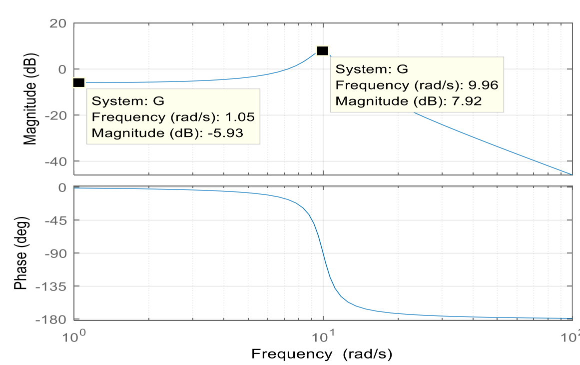

Figure 3 FRF associated with (3).

Straight and Curved Line Approximations of the Bode Plot

In days gone by engineers did not have immediate access to Bode plots. Those plots were often constructed by hand. The advent of advanced computing hardware and software has alleviated the need for such constructions. However, it has also tended to divorce the engineer from the qualitative behavior associated with an FRF. In fact, equations (8) were used routinely to gain quick and easy insight into the FRF of an underdamped system. We believe that there is no substitute for having qualitative insight into the properties of a transfer function that dictate the structure of the FRF. For this reason, we will now develop a few basic tools that one can use to approximate an FRF using a bare minimum of mathematics. We will proceed in the context of the above examples.

Example 1 (continued) Consider $G(s) = \frac{1}{s+1}$. Then imagine in your mind that $s = i\omega$. For $s << 1$ we have $G(s) \cong 1$. In other words, the magnitude and phase at low frequencies is $M \cong 0\ dB$ and $\theta \cong 0^o$. For $s >> 1$ we have $G(s) \cong 1/s$. Hence, The magnitude at s is: $M(s) = 20\log(1/s) = -20\log(s)$. The magnitude at $10s$ is:

$$M(10s) = 20\log(1/10s) = -20\log(10s) = -20\log(s) - 20 . \tag{9}$$

Hence, from (9) we see that for frequencies $s >> 1$, the magnitude drops by 20dB per decade of increasing frequency. In other words, the slope of the FRF magnitude is -20dB/decade. Clearly, since at such high frequencies we have $G(s) \cong 1/s \equiv 1/i\omega = -i(1/\omega)$, the high frequency phase is $\theta \cong -90^o$.

We will now define what constitutes as low and high frequencies in relation to a straight line approximate Bode plot.

Magnitude approximation: For $\omega < 1$, $M \cong 0\ dB$, and for $\omega > 1$, $M(\omega)$ has a slope of $-20\ dB/decade$.

Phase Approximation: For $\omega < 1/10$, $\theta(\omega) \cong 0^o$, and for $\omega > 10$, $\theta(\omega) \cong -90^o$. For $1/10 < \omega < 10$, $\theta(\omega)$ has a slope of $-45^o / decade$.

The straight line approximate Bode plot is shown at right. The magnitude approximation is quite good, except at the frequency $\omega = 1$, which corresponds to the pole at $s = -1$. At this frequency the true magnitude is $M(1) = 1/\sqrt{2}$, or $M(1)_{dB} \cong -3dB$. Hence, an improved curved line approximation is easily obtained by using a French curve to smooth out the sharp break at $\omega = 1$.

We see that the phase approximation is quite good at low and high frequencies. Moreover, it is exact at $\omega = 1$. Hence, we could use a French curve to smooth it out in a similar fashion.

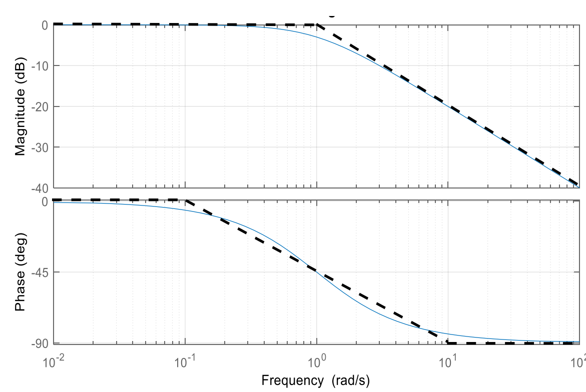

Figure 4 Straight line Bode approximation.

Example 2 (continued) We have $G(s) = \frac{50}{s^2 + 2s + 100}$, with $\zeta = 0.1$ and $\omega_n = 10$.

Magnitude approximation: For $\omega < 10$, $M \cong -6\ dB$, and for $\omega > 10$, $M(\omega)$ has a slope of $-40\ dB/decade$.

Phase Approximation: For $\omega < 1$, $\theta(\omega) \cong 0^o$, and for $\omega > 100$, $\theta(\omega) \cong -180^o$. For $1 < \omega < 100$, $\theta(\omega)$ has a slope of $-90^o / decade$.

The straight line approximation is shown at right. To obtain an improved curved line approximation of the magnitude we would use a French curve so that the approximation passes through the peak.

The straight line approximation of the phase excellent at very low and high frequencies (not shown), but is not so good in the region shown, except at $\omega \cong 10$ where it is excellent. Clearly, one would need to compute a few points in order to obtain a curved line approximation.

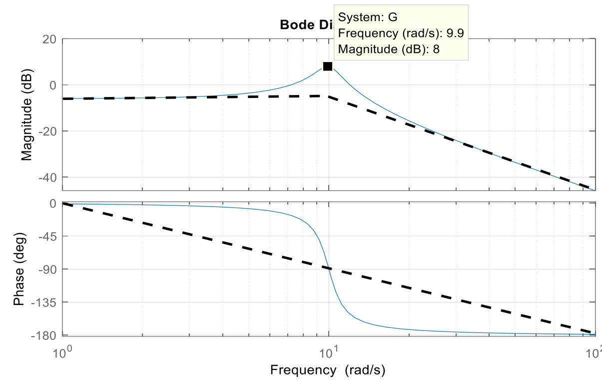

Figure 5 Straight line Bode approximation.

Some important conclusions

1. For a first order term $s+\omega_1$: The magnitude straight line break frequency is at ω_1. Above this frequency the slope is $\pm 20\,dB/decade$. The sign will + if this term is in the numerator and – if it is in the denominator. The phase straight line approximation has TWO break frequencies: One at $\omega_1/10$ and the other at $10\omega_1$. Below the lower break frequency the phase will be $\sim 0^o$. Above the higher break frequency it will be $\sim \pm 90^o$. The sign will + if this term is in the numerator and – if it is in the denominator.

2. For a second order term $s^2+2\zeta\omega_n s+\omega_n^2$ with $\zeta<1$: The magnitude straight line break frequency is at ω_n. Above this frequency the slope is $\pm 40\,dB/decade$. The sign will + if this term is in the numerator and – if it is in the denominator. The phase straight line approximation has TWO break frequencies: One at $\omega_n/10$ and the other at $10\omega_n$. Below the lower break frequency the phase will be $\sim 0^o$. Above the higher break frequency it will be $\sim \pm 180^o$. The sign will + if this term is in the numerator and – if it is in the denominator.

Example 3 To motivate the use of FRFs in relation to controller design, consider the controller $G_c(s)=\dfrac{s+\omega_z}{s+\omega_p}$. This controller has a zero at $-\omega_z$ and a pole at $-\omega_p$. It is comprised of two first order terms. Hence the FRF magnitude will have two break frequencies; one at ω_z and the other at ω_p. Suppose that $\omega_z<\omega_p$. Then the magnitude will be flat up to ω_z, at which point it will increase at 20 dB/decade. This increase will continue up to ω_p, at which point the denominator term will contribute a slope of -20 dB/decade, thereby canceling the +20 dB/decade. The low frequency magnitude is ω_z/ω_p, and the high frequency magnitude is one.

The above magnitude analysis was easy because there are only two break frequencies. The phase analysis is trickier since each term in $G_c(s)$ has two break frequencies. At both low and high frequencies the phase will be zero. For $\omega_z<\omega_p$ the phase of the numerator term will 'kick in' before that of the denominator term. This will result in increasing phase. Similarly, when the phase of the numerator term 'turns off', the phase of the denominator term will pull the controller phase back to zero.

To illustrate the above, let $G_c(s)=\dfrac{s+1}{s+20}$. The straight line magnitude is easy to see. The behavior of the phase, on the other hand, requires a bit of thought. □

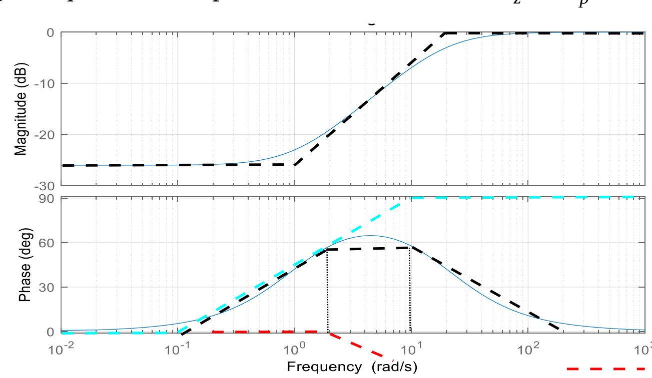

Figure 6 Straight line controller Bode plots.

The FRF of a 'Typical' Human Ear [https://www.soundonsound.com/sound-advice/how-ear-works]

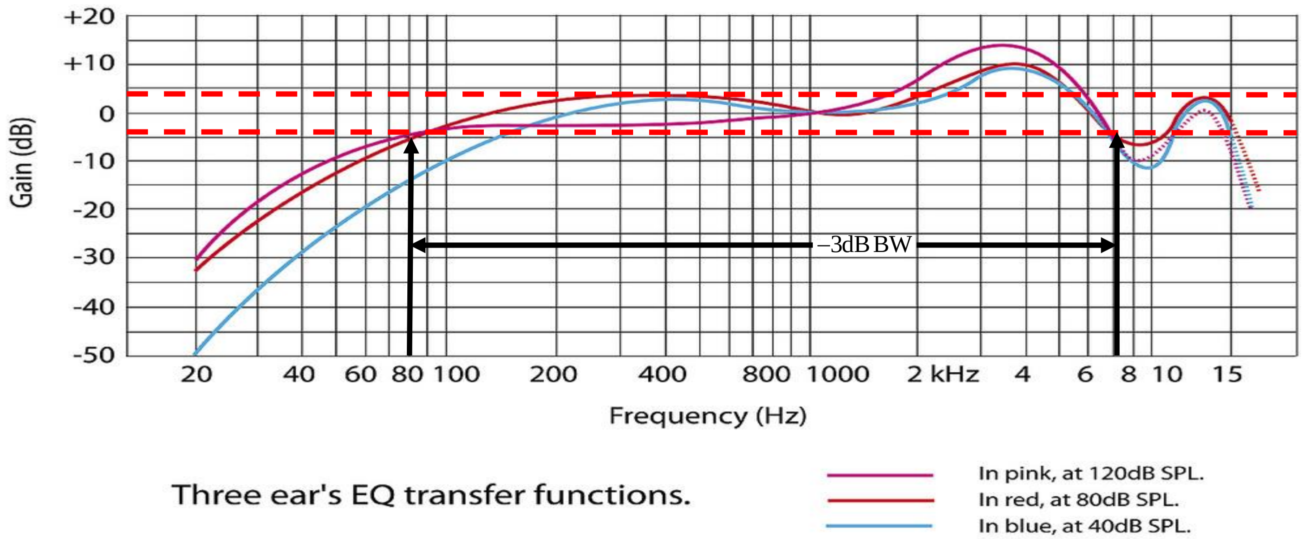

Figure 7. FRF's for a 'typical' human ear at three levels of loudness.

Observations and Comments-

1. The curves are termed transfer functions. This is incorrect. They are functions of ω, not *s*.

2. The figure includes three FRFs, corresponding to three equal loudness levels. This reflects the fact that the human ear is a nonlinear dynamical system.

3. The -3dB BW range of hearing is ~80Hz to 7.5kHz.

4. There is one resonance @ ~3.5kHz, and a second @ ~13.5kHz.