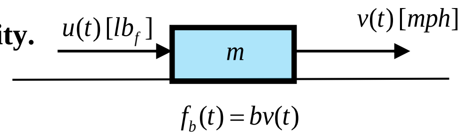
Lecture 1 An Introduction to First Order Linear Systems

***Example* 1. A simple model for automobile velocity.**



$$\sum_{forces} = m\dot{v} = u - bv$$
$$\Downarrow$$
$$\boxed{m\dot{v}(t) + bv(t) = u(t)\,;\; v(0\_) = v_o}$$

Solution for a *step input*: Suppose that $u(t) = U_o\, lb_f$ for $t > 0$.

***<u>Method 1</u>* ("Assumed Solution Technique"):**

<u>Part I</u>: The complementary solution to the homogeneous equation: For the *homogeneous* equation $m\dot{v} + bv = 0$, assume a *complementary* solution of the form $v_c(t) = Ce^{st}$, where C and s are parameters to be found. It follows that $\dot{v}_c(t) = Cse^{st}$. Substituting these into the homogeneous equation gives

$$Cmse^{st} + Cbe^{st} \;=\; (ms+b)Ce^{st} \;=\; 0 \qquad (1)$$

Since (1) must hold for any time, t, it follows that we must have either $C = 0$ (the trivial solution) or $ms + b = 0$. Since we are not interested in the trivial solution, $s = -b/m$. Hence, the complementary solution is:

$$v_c(t) \;=\; Ce^{-(b/m)t}. \qquad (2)$$

<u>Part II</u>: The particular solution. Assume that the particular solution is a linear combination of the form of $u(t)$ and all of its derivatives. Since $u(t) = U_o\, lb_f$, all of its derivatives equal zero. Hence, the particular solution has the form $v_p(t) = V_o$. Substituting this into the differential equation gives: $bV_o = U_o \;\Rightarrow\; V_o = U_o/b$. (3)

The system *static gain* is the ratio of the constant output over the constant input. In this case, it is $g_s = V_o/U_o = 1/b$.

<u>Part III</u>: The total solution. The total solution is the sum of the complementary and particular solutions:

$$v(t) \;=\; Ce^{-(b/m)t} \;+\; U_o/b. \qquad (4)$$

Finally, for an initial condition, we can obtain the value of the parameter, C:

$$v(0) = v_o = C + 100/b \;\Rightarrow\; C = v_o - U_o/b. \qquad (5)$$

Substituting (5) into (4), and rearranging terms, gives the solution:

$$v(t) \;=\; v_o e^{-(b/m)t} \;+\; \frac{U_o}{b}(1 - e^{-(b/m)t}). \qquad (6)$$

Notice that the velocity entails two parts: one that results from the initial condition, and the other that results from the applied force. Also, notice that the term $e^{-(b/m)t}$ goes to zero as $t \to +\infty$ since both b and m are positive. Define the parameter $\tau \overset{\Delta}{=} m/b$, then we can plot this term where t is in terms of multiples of τ:

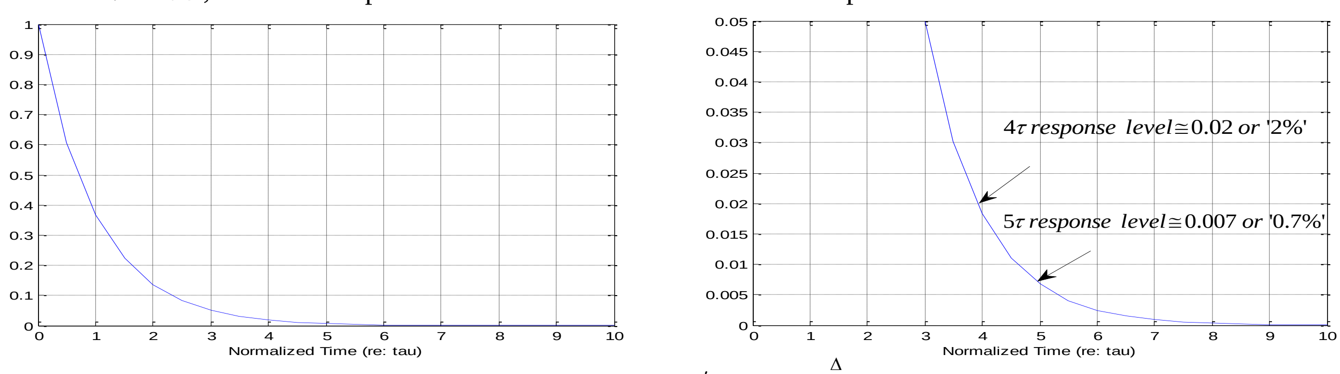


Figure 1. Plot of $e^{-t'}$ versus $t' \overset{\Delta}{=} t/\tau$.

The solution (6) is characterized by two quantities:

(i) The system ***time constant***, $\tau = m/b$: For times $t \geq 4\tau$ (or, $t \geq 5\tau$ if you choose), the exponential term become negligible, and the response is said to be in the ***steady state***.

(ii) The system ***static gain***, $g_s = V_o / U_o = 1/b\,[mph/lb_f]$. This is simply the ratio of the *steady state* output to the input.

Definition 1. The function of time, 1(*t*), which equals 1 for $t \geq 0$ and zero for $t < 0$ is called the ***unit step*** function.

The general form then, for a first order system response, *y*(*t*), to an initial condition, y_o and ***characteristic polynomial*** $P(s) = s + 1/\tau$, to a *step* input $f(t) = F_o \bullet 1(t)$ is:

$$v(t) \quad = \quad v_o e^{-t/\tau} \quad + \quad g_s F_o (1 - e^{-t/\tau}). \tag{7}$$

A Brief Introduction to the Laplace Transform

In a moment, we will proceed to use the method of *Laplace Transforms* to obtain the solution (7) of the differential equation:

$$m\dot{v}(t) + bv(t) = U_o 1(t)\,;\, v(0_-) = v_o. \tag{8}$$

First, let's use the definition of the *Laplace Transform* to see what (7) looks like in the *s*-domain.

Definition 2. Let $s = \sigma + j\omega$ be any complex number. The *Laplace transform* of a function *x*(*t*) that equals zero for *t*<0 (i.e. the *one-sided Laplace transform*) is defined as:

$$\ell[x](s) = \int_{t=0}^{\infty} x(t) e^{-st} dt \overset{\Delta}{=} X(s). \tag{9}$$

Now, let's use (9) to compute the Laplace transform of the functions of time involved in (8).

***Laplace transform of the unit step*:**

$$\ell[1](s) = \int_{t=0}^{\infty} e^{-st} dt = \left.\frac{e^{-st}}{-s}\right|_{t=0}^{\infty} = \left.\frac{e^{-st}}{s}\right|_{t=\infty}^{0} = \frac{1}{s}(1 - \lim_{t\to\infty} e^{-st}).$$

Substituting $s = \sigma + j\omega$ into the limit term gives

$$\lim_{t\to\infty} e^{-st} = \lim_{t\to\infty} e^{-(\sigma + j\omega)t} = \lim_{t\to\infty} e^{-\sigma t} \bullet e^{j\omega t} = 0 \; if \; and \; only \, if \; \sigma > 0.$$

Hence, we obtain

$$\boxed{\ell[1](s) = \frac{1}{s} \text{ for all } s = \sigma + j\omega \text{ such that } \sigma > 0.}$$

Laplace transform of the decaying exponential $x(t) = e^{-at}$ **:**

$$\ell[x](s) = \int_{t=0}^{\infty} e^{-at} e^{-st} dt = \int_{t=0}^{\infty} e^{-(s+a)t} dt = \int_{t=0}^{\infty} e^{-s't} dt \quad \textit{where} \quad s' \stackrel{\Delta}{=} s + a\,.$$

This is exactly the Laplace transform of the unit step function, but for $s' \stackrel{\Delta}{=} s + a$. Hence, we obtain

$$\ell(e^{-at}) = \frac{1}{s'} \text{ for all } s' = \sigma + j\omega \text{ such that } \sigma > 0\text{, where } s' \stackrel{\Delta}{=} s + a\,.$$

In other words,

$$\boxed{\ell(e^{-at}) = \frac{1}{s+a} \text{ for all } s = \sigma + j\omega \text{ such that } \sigma > -a\,.}$$

Before we can apply the two boxed Laplace transforms to (7), we need to recognize that the Laplace operation $\ell[\bullet](s)$ is a ***linear operation***; that is: $\ell[c_1 x_1 + c_2 x_2](s) \;=\; c_1\ell[x_1](s) + c_2\ell[x_2](s)$. That this holds follows directly from the fact that the integral of the sum of two integrable functions is the sum of the integrals, and that a multiplying constant can always be brought outside of the integral.

Now, let's take the Laplace transform of (7): $V(s) \stackrel{\Delta}{=} \ell[v](s) \;=\; \dfrac{v_o}{s + (1/\tau)} \;+\; g_s U_o \left[\dfrac{1}{s} - \dfrac{1}{s + (1/\tau)}\right]$.

However, notice that the input, $u(t) = U_o 1(t)$ has Laplace transform $U(s) = U_o / s$. Hence, we have

$$V(s) = \frac{v_o}{s + (1/\tau)} + \left[\frac{g_s}{\tau s + 1}\right] U(s) \stackrel{\Delta}{=} \frac{v_o}{s + (1/\tau)} + W(s)U(s) \tag{10}$$

where we have defined $W(s) \stackrel{\Delta}{=} \dfrac{g_s}{\tau s + 1}$. This function of *s* has special importance; for if the initial condition were $v_o = 0$, we would then have $W(s) = V(s)/U(s)$. In words, the ratio of the Laplace transform of the system output and the system input <u>*under zero initial conditions*</u> is called the system ***transfer function***.

We will show that (10) holds not only for a step input, but for *any* input $u(t)$ with Laplace transform $U(s)$. For now, the important thing to observe is that (7) and its Laplace transform (10) are composed of two parts: that associated with the initial condition, and that associated with the input.

QUESTION: If we already have the solution (7), what was the point of obtaining (10)?

ANSWER: Firstly, we used the method of Laplace transforms; not the 'standard' method. Second, (10) is more general than (7), in that it is the solution of the O.D.E. in the *s*-domain for *any* input $u(t) \leftrightarrow U(s)$. Third, it resulted in the definition of a *transfer function* that relates any input $u(t)$ to the resulting output $y(t)$. Finally, having the system transfer function allows us to immediately obtain the system *frequency response function* (FRF), which is a ubiquitous entity in engineering systems.

To arrive at the definition of a system FRF, we begin with

Definition 3. Let $s = j\omega$ be any purely imaginary number. Then the *Laplace transform* of a function *x*(*t*) is called the ***Fourier transform***:

$$\ell[x](s=j\omega)\overset{\Delta}{=}\Im[x](j\omega)=\int_{t=0}^{\infty}x(t)e^{-j\omega t}dt\overset{\Delta}{=}X(j\omega)\,. \qquad (11)$$

The Fourier transform of a function of time is extremely important and is commonly addressed in research and development spheres. It gives a description of the frequency structure of $x(t)$. This is central to what in electrical engineering is called *filter theory*. The Fourier transform of (7) is:

$$V(j\omega)=\frac{v_o\tau}{j\omega\tau+1}+W(j\omega)U(j\omega)=W(j\omega)\bullet\left[U(j\omega)+\frac{v_o\tau}{g_s}\right]. \qquad (12)$$

Moreover, the quantity $W(s=i\omega)\overset{\Delta}{=}\dfrac{g_s}{1+i(\tau\omega)}$ is defined to be the system FRF.

***Method* 2: Using a Table of Laplace Transform Pairs (i.e. a 'short cut' :)**

Let's recall again the differential equation, (8), that we desire to solve: $m\dot{v}(t)+bv(t)=U_o1(t)\,;\,v(0\_)=v_o$. In view of the discussion of the static gain and time constant, we will rewrite this as:

$$\tau\dot{v}(t)+v(t)=g_su(t)\,;\,v(0\_)=v_o \qquad (13a)$$

where $$\tau\overset{\Delta}{=}\frac{m}{b}\ ,\quad g_s\overset{\Delta}{=}\frac{1}{b}\ ,\text{ and } u(t)=U_o1(t)\,. \qquad (13b)$$

The Most Important Formula for Using Laplace Transforms: The Derivatives of a Function of Time, $x(t)$:

We require the following result from integral calculus, known as *integration by parts*: $\int u\,dv=uv-\int v\,du$. (14)

Recall from (9) that the Laplace transform of $x(t)$ is: $\ell[x](s)=\int_{t=0}^{\infty}x(t)e^{-st}dt\overset{\Delta}{=}X(s)$

To apply (14) to this, let $u=x(t)$ and $dv=e^{-st}dt$. Then $v=-\dfrac{1}{s}e^{-st}$ and $du=\dot{x}(t)\,dt$. We then have

$$X(s)=-\frac{x(t)}{s}e^{-st}\Big|_{t=0\_}^{\infty}+\frac{1}{s}\int_{0\_}^{\infty}\dot{x}(t)\,e^{-st}dt\quad=\quad\frac{x(0\_)}{s}+\frac{1}{s}\,\ell[\dot{x}](s)\,. \qquad (15)$$

Hence, we arrive at: $$\ell[\dot{x}](s)=s\,X(s)-x(0\_)\,. \qquad (16)$$

Equation (16) is, perhaps, the single most important equation in relation to solving O.D.E.s via Laplace transforms. To convince the reader of that, let's obtain a similar relation for higher derivatives. To begin, define $g(t)\overset{\Delta}{=}\dot{x}(t)$. Then, applying (16) to the derivative of $g(t)$ gives:

$$\ell[\dot{g}](s)=s\,G(s)-g(0\_)\,. \qquad (17a)$$

However, $G(s)=s\,X(s)-x(0\_)$, and $\dot{g}(t)=\ddot{x}(t)$. And so we have

$$\ell[\ddot{x}](s)=s^2\,X(s)-s\,x(0\_)-\dot{x}(0\_) \qquad (17b)$$

Repeating this for $\dddot{x}(t)$, it is easy to see that we get: $\ell[\dddot{x}](s) = s^3 X(s) - s^2 x(0\_) - s\dot{x}(0\_) - \ddot{x}(0\_)$. (17c)

It is worth mentioning why the derivation of (17) was given. After all, this is not a math course. It was given for three reasons. First, it demonstrates the value of results from calculus. Second, by understanding how the result is arrived at, the student may have more confidence in using it. And third, it highlights the ease with which the Laplace transform of higher derivatives can be obtained when needed.

QUESTION: Why is (16) [and, consequently (17)] so important?

ANSWER: Because if we assume zero initial conditions (an assumption needed to define a transfer function), then to obtain the transfer function we simply replace the kth derivative of any quantity x(t) by $s^k X(s)$. In other words: ***we convert a differential equation into an algebraic equation.***

We are now in a position to solve (13) via the *Method of Laplace Transforms*. Recall that $\ell[*]$ is a *linear operation*. Hence: $\ell[\tau\dot{v} + v] = \tau\ell[\dot{v}] + \ell[v] = g_s\ell[u]$, or, $\tau[sV(s) - v_o] + V(s) = g_s U(s)$.

Solving this for $V(s)$ gives: $V(s) = \dfrac{v_o}{s + (1/\tau)} + \left[\dfrac{g_s}{\tau s + 1}\right]U(s) \stackrel{\Delta}{=} \dfrac{v_o}{s + (1/\tau)} + W(s)U(s)$ (18)

where $W(s) \stackrel{\Delta}{=} \dfrac{g_s}{\tau s + 1}$. This is exactly equation (10) that was obtained via the assumed solution method.

Under ***<u>zero initial conditions</u>***, (18) yields the system ***transfer function*** $W(s) = \dfrac{V(s)}{U(s)} = \dfrac{g_s}{\tau s + 1}$

<u>This is where the real short cut comes into play:</u> To obtain the time domain solution, we need to take the *inverse Laplace transform* of (18). To this end, we will ‘cop out’ of the math world, and simply use the table of Laplace transforms on the inside of the front cover of the book. It is a fact that the inverse Laplace transform is also a *linear operation.* Hence, because the right side of (18) is the sum of two terms, we will need two entries from the table.

The term $\dfrac{v_o}{s + (1/\tau)}$ resembles entry #7: $\dfrac{1}{s + a} \leftrightarrow e^{-at}$. Hence,

$$\ell^{-1}\left[\frac{v_o}{s + (1/\tau)}\right] = v_o e^{-t/\tau} \quad . \qquad (19a)$$

We already developed entry #2; namely $U(s) = \dfrac{U_o}{s}$, and so the second term on the right side of (18) is:

$$W(s)U(s) = \frac{g_s U_o}{s(\tau s + 1)} .$$

This term resembles entry #11: $\dfrac{a}{s(s + a)} \leftrightarrow 1 - e^{-at}$. The trick now is to get the term into a constant times the form of this entry: $W(s)U(s) = \dfrac{g_s U_o}{s(\tau s + 1)} = \dfrac{g_s U_o/\tau}{s(s + 1/\tau)} = g_s U_o \bullet \dfrac{1/\tau}{s(s + 1/\tau)}$. And so, we arrive at:

$$\ell^{-1}[W(s)U(s)] = g_s U_o \bullet \ell^{-1}\left[\frac{1/\tau}{s(s + 1/\tau)}\right] = g_s U_o \bullet (1 - e^{-t/\tau}) . \qquad (19b)$$

From (19), it follows that the inverse Laplace transform of (18) is

$$v(t) \;=\; v_o e^{-t/\tau} + g_s U_o (1 - e^{-t/\tau})\,. \tag{20}$$

Notice that (6) and (20) are identical.

Discussion

We have presented two methods for solving a first order differential equation with constant coefficients. The assumed solution method relies on the form of the assumed solutions; both for the complimentary and the particular solutions. These solutions must then be combined prior to using the given initial condition to solve for the unknown constant in the complimentary solution. This is the direct method of solution presented in a first course in calculus. The Laplace transform method relies almost entirely upon use of a table of transforms. Some tables (though not the one in our text) even include the transform relations for time derivatives that we derived here.

Whether or not one prefers one method over the other, the fact is that we will be working in the *s*-domain almost exclusively throughout the course. It is in this domain that the important concept of the system *transfer function* is cast. It cannot be emphasized enough that formulation of a system transfer function from a differential equation requires that *all initial conditions are presumed to be zero.*

Before we proceed to address second order linear systems, let's end this discussion with a quick look at an n^{th} order linear system. Specifically, consider the system described by the following n^{th} order differential equation:

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t) = b_m f^{(m)}(t) + b_{m-1} f^{(m-1)}(t) + \cdots + b_1 f^{(1)}(t) + b_0 f(t)$$

The student may have been exposed to this type of rather 'ugly' equation in a prior course involving the *state space* method of solution. We will address that method later in the course. For now, let's presume that all initial conditions are zero. Then, taking the Laplace transform of this equation yields the system transfer function:

$$W(s) = \frac{Y(s)}{F(s)} = \frac{b_m s^m + b_{m-1} s^{(m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \overset{\Delta}{=} \frac{B(s)}{A(s)}\,.$$

This is called a *rational* transfer function because it is the ratio of two polynomials in *s*. The denominator polynomial, $A(s)$ is the *characteristic polynomial* associated with the complimentary portion of the assumed solution method. Recall that the form of the assumed solution is $C e^{st}$, and that the values of *s* that work are the roots of this polynomial. If even one of the roots of $A(s)$ has a positive real part [i.e. is in the Right Half Plane (RHP)], then that solution will go to infinity as *t* does. This is not a good thing! Hence, in order for a system to be 'well-behaved' or 'stable' we require that all the roots of the characteristic polynomial be in the LHP.

This awareness that system stability requires that all roots of the characteristic polynomial be in the LHP was arrived at almost trivially as a result of looking at the system transfer function and its relation to the assumed solution method. No linear algebra was required.

A numerical example- Consider the equation: $2\dot{y} + 5y = 10u(t)$

(a) Give the system transfer function.

<u>*Solution*</u>: $2sY(s) + 5Y(s) = 10U(s) \;\;\Rightarrow\;\; G(s) \overset{\Delta}{=} \frac{Y(s)}{U(s)} = \frac{10}{2s+5}\,.$

(b) Compute the magnitude and phase of the FRF.

<u>*Solution*</u>: $G(i\omega) = \frac{10}{i2\omega + 5} = \frac{10}{\sqrt{5^2 + (2\omega)^2}} e^{i\theta(\omega)} = M(\omega) e^{i\theta(\omega)}$ where $M(\omega) = \frac{10}{\sqrt{5^2 + (2\omega)^2}}$ and $\theta(\omega) = -a\tan(2\omega/5)\,.$

(c) Find the system time constant and static gain.

<u>*Solution*</u>: $G(s)=\dfrac{10}{2s+5}=\dfrac{2}{0.4s+1}=\dfrac{g_s}{\tau s+1}$.

(d) Obtain the response to a unit *ramp* input.

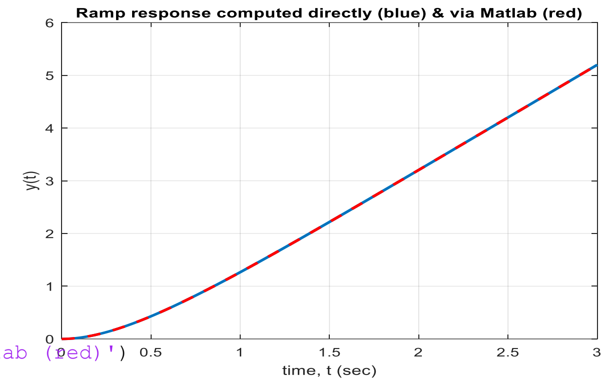
<u>*Solution*</u>: $Y(s)=G(s)U(s)=\left(\dfrac{10}{2s+5}\right)\left(\dfrac{1}{s^2}\right)=\dfrac{5}{s^2(s+2.5)}=\left(\dfrac{5}{2.5}\right)\left(\dfrac{2.5}{s^2(s+2.5)}\right)$.

Using the Laplace table entry #12 gives: $y(t)=\left(\dfrac{2}{2.5}\right)\left[2.5t-1+e^{-2.5t}\right]$

(e) Plot the response in (c). Then use Matlab commands to verify its correctness.

<u>*Solution*</u>:

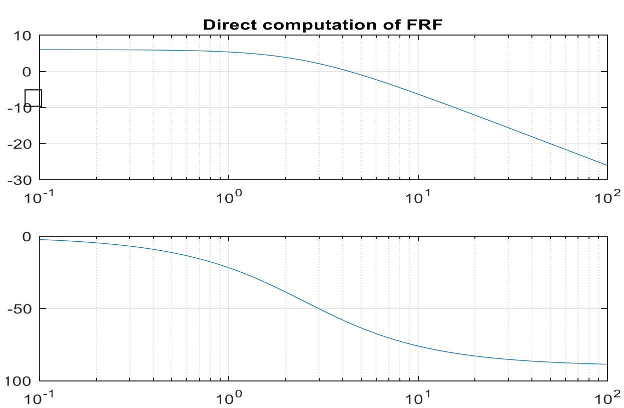
```
%PROGRAM NAME: Lecture2.m
%Compute ramp response directly:
t=0:.001:3;
y=0.8*(2.5*t - 1 + exp(-2.5*t));
figure(1)
plot(t,y,'LineWidth',2)
grid
hold on
%Verify using Matlab tf:
G=tf(10,[2,5]); %Transfer function
r=t; %Unit ramp input
yy=lsim(G,r,t);
plot(t,yy,'r--','LineWidth',2)
title('Ramp response computed directly (blue) & via Matlab (red)')
xlabel('time, t (sec)')
ylabel('y(t)')
```

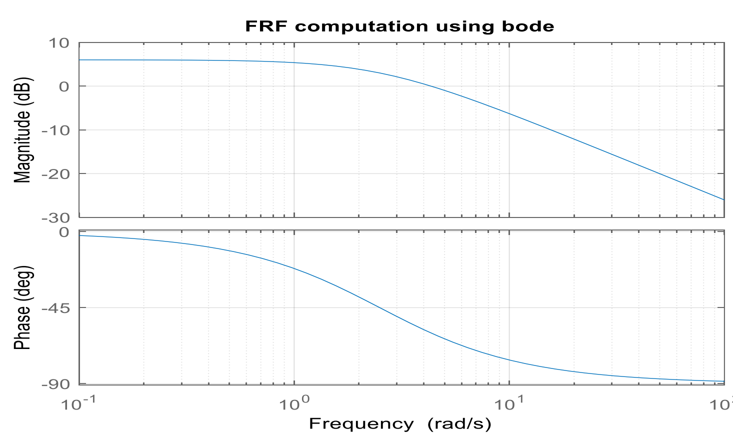


(f) Compute and plot the system FRF directly (convert magnitude to dB). Then verify correctness by using the using the Matlab ‘bode’ command.

<u>*Solution*</u>:

```
%Compute FRF:
w=0.1:.001:100;
M=10*(5^2 +(2*w).^2).^-0.5;
MdB=20*log10(M);
th=-atand(2*w/5);
figure(2)
subplot(2,1,1),semilogx(w,MdB)
hold on
grid
title('Direct computation of FRF')
subplot(2,1,2),semilogx(w,th)
hold on
grid
figure(3)
bode(G)
grid
title('FRF computation using bode')
```





End of Lecture 1