**Homework 5 AERE331 Spring 2020 Due 4/22(W) SOLUTION**

**PROBLEM 1(25pts)** This problem addresses a research topic that is receiving renewed attention in recent years. It relates to vortex shedding turbulence off the trailing edge of an airfoil, as shown in Figure 3 below; taken from [http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.569.7015&rep=rep1&type=pdf](http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.569.7015&rep=rep1&type=pdf) . The article was published in *J. Fluid Mech.*(2009),vol.632,pp.245–271.

As can be seen from plate (b), the turbulence off the trailing edge is of an oscillating nature. This can lead to flapping of the trailing edge, as well as to oscillatory downwash. In this problem we will arrive at a model to simulate such turbulence.

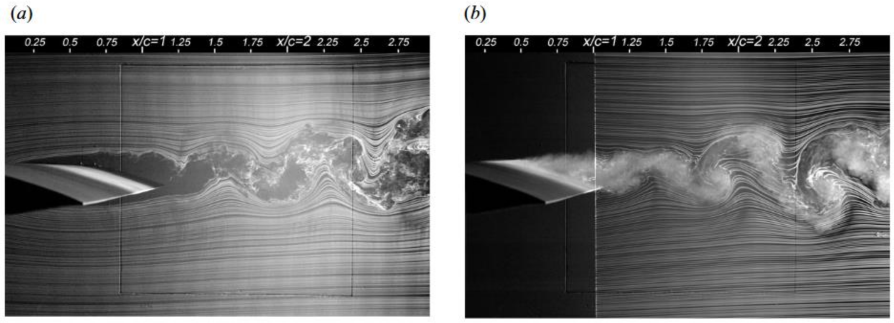


FIGURE 3. Flow visualization for $Re_c = 100 \times 10^3$ at $\alpha = 5°$: ($a$) upstream smoke wire; ($b$) downstream smoke wire.

**(a)(5pts)** To accommodate the oscillating nature of the turbulence, consider the shaping filter: $G(s) = \dfrac{c}{s^2 + 0.2s + 100}$ .

Use the Matlab 'integral' command to find the value of c so that $\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|^2 \, d\omega = 1$. Give **ALL** of your code **HERE**.

<u>*Solution*</u>: `g=@(w)abs((1i*w).^2+0.2*(1i*w)+100).^-2;gint=(1/pi)*integral(g,0,inf)=`**`.025`**
Hence, $c^2 = 1/.025 = 40$, which gives $c = \mathbf{6.3246}$.

**(b)(10pts)** Regardless of your answer in (a), assume here that $c = \mathbf{6.3}$. Suppose that we require that the turbulence have power $\sigma^2 = 25$. Arrive at the turbulence power spectral density plot.
NOTE: Use w=logspace(0,2,5000) and give $S(\omega)$ in dB.
<u>*Solution*</u>: [See code @ 2(b).]

This requires $G(s) = \dfrac{6.3(5)}{s^2 + 0.2s + 100}$. Hence,

$$S(\omega) = \frac{31.5^2}{|100 - \omega^2 + i0.2\omega|^2}.$$

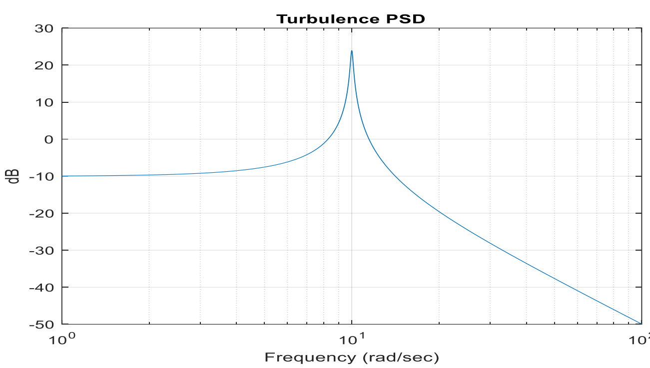


**Figure 2(b)** Plot of $S(\omega)$.

**(c)(10pts)** If we assume a Nyquist frequency $\omega_N = 1000\,rad/\sec$, the corresponding sampling period is $\Delta = \pi/\omega_N = 0.00314\sec$. Obtain a simulation of the turbulence over a 200-second window. Then discuss whether or not you believe the amplitudes are reasonable.
<u>*Solution*</u>: [See code @ 2(c).]

Since $\sigma = 5$, we would expect that $\pm 3\sigma = 15$. The plot supports this.

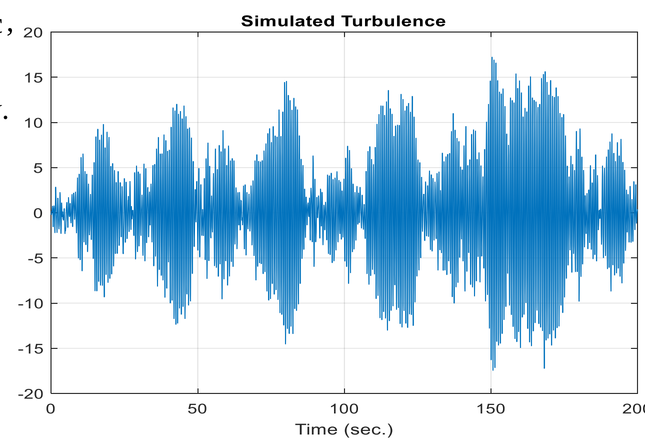


**Figure 3(c)** Plot of simulated turbulence.

**PROBLEM 2(40pts)** Consider the function $f(t) = e^{-t}$. This function could be a system impulse response. It could also be an input to the system.

**(a)(5pts)** (i) Show that $F(s) = \int_{t=0}^{\infty} f(t)e^{-st}dt = \frac{1}{s+1}$. Show all steps. Verify that the s-values of the complex plane for which the result holds includes the imaginary axis.

*Solution*:

(i): $F(s) = \int_{t=0}^{\infty} e^{-t}e^{-st}dt = \int_{t=0}^{\infty} e^{-(s+1)t}dt = \left.\frac{-e^{-(s+1)t}}{s+1}\right|_{t=0}^{\infty} = \lim_{t\to\infty}\frac{-e^{-(s+1)t}}{s+1} - \frac{-1}{s+1} = 0 - \frac{-1}{s+1} = \frac{1}{s+1}$.

(ii): The result assumes that $\lim_{t\to\infty} e^{-(s+1)t} = \lim_{t\to\infty} e^{-(\sigma+i\omega+1)t} = \lim_{t\to\infty} e^{-(\sigma+1)t}e^{-i\omega t} = 0$. This can only happen for $s = \sigma + i\omega$ with $s = \sigma > -1$. Hence, it holds for $s = i\omega$.

**(b)(4pts)** (i) Use the 'bode' command to plot the magnitude and phase of $F(i\omega)$. On the plot identify (ii) the -3dB BW frequency, and (iii) The frequency at which the magnitude is 30dB below the static gain.

*Solution*: [See code @ 2(b).]

(i): $\omega_{-3dB} = 1 rad/s$ (ii): $\omega_{-30dB} = 31.5 rad/s$

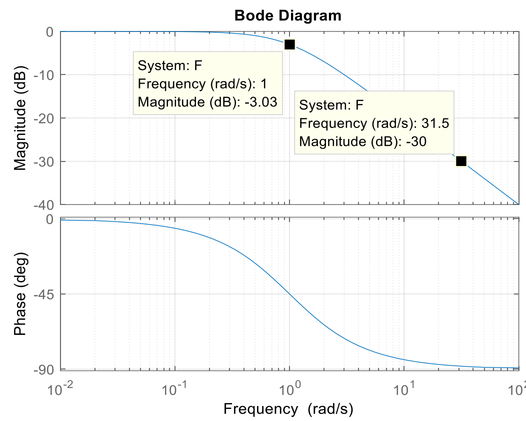


**Figure 2(b)** Bode plot of $F(s)$.

**(c)(5pts)** Suppose that we sample $f(t)$ using a sampling interval $T$. Then $f(kT) = e^{-kT}$. The numerical approximation of the Laplace transform is: $\hat{F}(s) = \sum_{k=0}^{\infty} f(kT)e^{-s(kT)}T$. Now consider the following

FACT 1: $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$. Proof: $(1-x)\left(\sum_{k=0}^{n} x^k\right)(1-x)(1+x+x^2+\cdots+x^n) = 1-x^{n+1}$ (i.e. all middle terms cancel.)

Use this fact to show that $\hat{F}(s) = \frac{T}{1-\alpha z^{-1}} \overset{\Delta}{=} \hat{F}(z)$, where we have defined $\alpha = e^{-T}$ and $z^{-1} = e^{-sT}$.

*Solution*:

$\hat{F}(s) = \sum_{k=0}^{\infty} e^{-kT}e^{-s(kT)}T = T\sum_{k=0}^{\infty} e^{[-(s+1)T]k} = T\lim_{k\to\infty}\sum_{k=0}^{n}(\alpha z^{-1})^k = T\lim_{k\to\infty}\frac{1-(\alpha z^{-1})^{n+1}}{1-\alpha z^{-1}} = \frac{T}{1-\alpha z^{-1}}$.

**(d)(3pts)** In order to arrive at the result in (c) you should have realized that it was required that $|\alpha z^{-1}| < 1$. Hence, (c) is only defined for $|z| > |\alpha|$. Let $\hat{F}(s) = \frac{T}{1-\alpha z^{-1}} \overset{\Delta}{=} \hat{F}(z)$ be a system transfer function. Then the associated FRF is $\hat{F}(s = i\omega) = \frac{T}{1-\alpha z^{-1}} \overset{\Delta}{=} \hat{F}(z = e^{i\omega T})$. Assuming this FRF is well-defined, show that the sampled system pole $z_1 = \alpha$ must satisfy $|\alpha| < 1$. Note: Do NOT revert to the s-domain. Use only the above information.

*Solution*:

We are told that $\hat{F}(z)$ requires that $|z|>|\alpha|$. We are also told that for $\hat{F}(z=e^{i\omega T})$. This means it is well-defined for $|z|=|e^{i\omega T}|=1$. Hence, we must have $|\alpha|<|e^{i\omega T}|=1$.

**(e)(3pts)** Recall that for <u>any</u> stable transfer function $F(s)$ the FRF $F(i\omega)$ is defined over the interval $-\infty<\omega<\infty$. Hence, the same is true for $\hat{F}(z=e^{i\omega T})$. However, $\hat{F}(z=e^{i\omega T})$ is a periodic function of $\omega$. Show that the period is $\omega_s \overset{\Delta}{=} 2\pi/T$. [In other words, show that for any stable $F(s)$ we have $\hat{F}(e^{i(\omega+\omega_s)T})=\hat{F}(e^{i\omega T})$.]

*Solution*:

$\hat{F}(e^{-i(\omega+\omega_s)T})=\hat{F}(e^{-i\omega T}\,e^{-i\omega_s T})=\hat{F}(e^{-i\omega T}\,e^{-i(2\pi/T)T})=\hat{F}(e^{-i\omega T}\,e^{-i2\pi})=\hat{F}(e^{-i\omega T})$.

**(f)(5pts)** The period $\omega_s \overset{\Delta}{=} 2\pi/T$ is called the radial ***sampling frequency*** since it is $2\pi$ over the sampling period *T*. From (e) it follows that $\hat{F}(z=e^{i\omega T})$ is uniquely defined over the frequency range $-\omega_N \le \omega < \omega_N$ where $\omega_N \overset{\Delta}{=} 0.5\omega_s$. The frequency $\omega_N \overset{\Delta}{=} 0.5\omega_s$ is called the radial ***Nyquist frequency***. In relation to $F(s)=\dfrac{1}{s+1}$ in (a), from (c) we have $\hat{F}(z)=\dfrac{T}{1-\alpha z^{-1}}$, where $\alpha=e^{-T}$ and $z^{-1}=e^{-sT}$. In relation to $\hat{F}(e^{i\omega T})$, show that its magnitude is $M(\omega)=\dfrac{T}{\sqrt{1+\alpha^2-2\alpha\cos(\omega T)}}$ and its phase is $\theta(\omega)=-\tan^{-1}\left(\dfrac{\sin(\omega T)}{1-\alpha\cos(\omega T)}\right)$.

*Solution*: $\hat{F}(e^{i\omega T})=\dfrac{T}{1-\alpha e^{-i\omega T}}=\dfrac{T}{1-\alpha[\cos(\omega T)-i\sin(\omega T)]}=\dfrac{T}{[1-\alpha\cos(\omega T)]+i\sin(\omega T)}$. (*)

Hence, $M(\omega)=\dfrac{T}{\sqrt{[1-\alpha\cos(\omega T)]^2+[\alpha\sin(\omega T)]^2}}=\dfrac{T}{\sqrt{[1-2\alpha\cos(\omega T)+\alpha^2\cos^2(\omega T)]+\alpha^2\sin^2(\omega T)}}=\dfrac{T}{\sqrt{1+\alpha^2-2\alpha\cos(\omega T)}}$.

From (*) it is clear that $\theta(\omega)=-\tan^{-1}\left(\dfrac{\sin(\omega T)}{1-\alpha\cos(\omega T)}\right)$.

**(g)(5pts)** In (a) you should have found that $\omega_{-30dB}=31.5\,rad/s$. Let $\omega_N=\omega_{-30dB}$. Overlay plots of (i) the magnitudes (dB) and (ii) the phases (deg.) of $F(i\omega)$ and $\hat{F}(z=e^{i\omega T})$. Use a log-spaced frequency axis for $0.01\le\omega<\omega_N$.

*Solution*: [See code @ 2(g).]

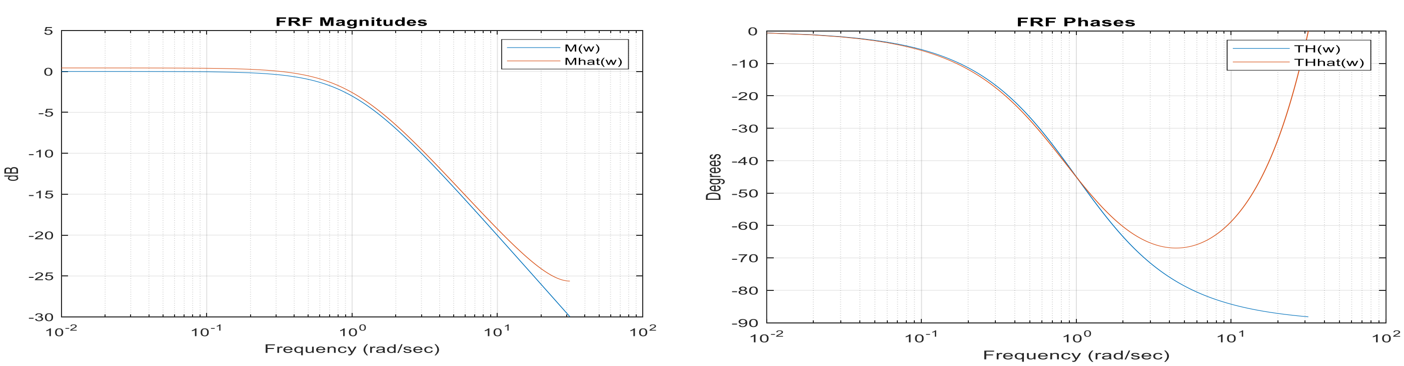


**Figure 2(g).** FRF magnitudes (LEFT) and phases (RIGHT).

**PROBLEM 3(20pts)** This problem is a continuation of PROBLEM 2. In this problem we will view $F(s)$ as a transfer function.

**(a)(8pts)** From 2(c) $\hat{F}(z) = \frac{Tz}{z-\alpha}$ is a ratio of polynomials in $z$.

Use the 'bode' command to obtain overlaid plots of $F(s)$ and $\hat{F}(z)$. Then comment as to whether or not they validate your plots in 1(g).

*Solution*: [Give your code HERE.]

```
%(a):
figure(30)
bode(F)
hold on
Fhat=tf([T 0],[1 -A],T);
bode(Fhat)
legend('F','Fhat')
grid
```

Yes, they validate my former plots.

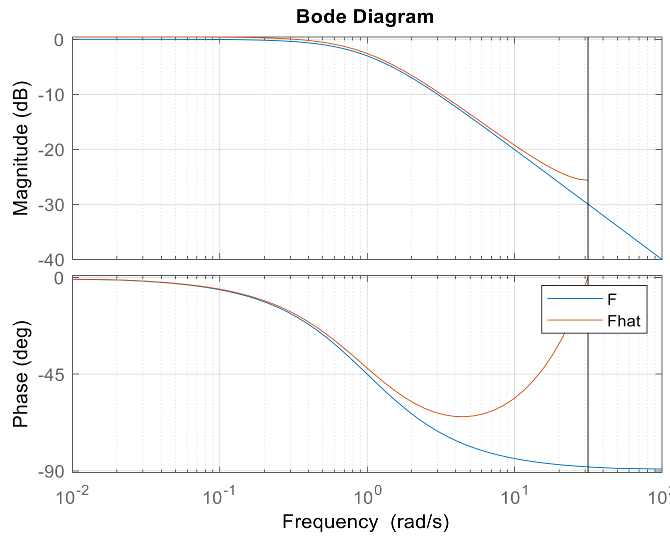


**Figure 3(a)** Bode plots of $F(s)$ and $\hat{F}(z)$.

**(b)(7pts)** Use the 'step' command to overlay the associated unit step responses. Then comment on how they compare.

*Solution*: [Give your code HERE.]

```
figure(31)
step(F)
hold on
step(Fhat)
legend('y(t)','yhat(t)')
title('Unit Step Responses')
grid
```

With the exception of the different ss values they compare reasonably well.

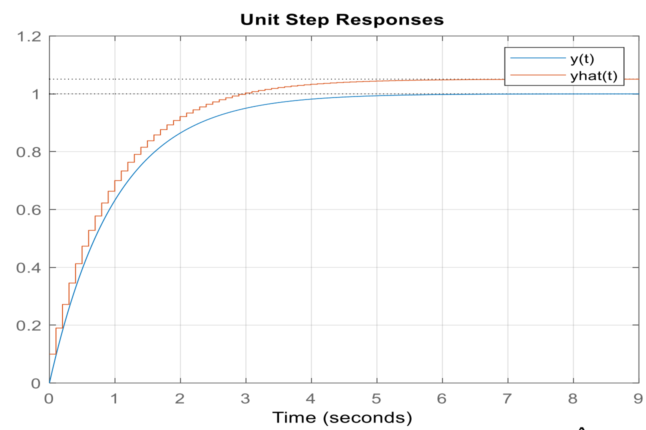


**Figure 3(b)** Step responses for $F(s)$ and $\hat{F}(z)$.

**(c)(5pts)** In (b) you should have found that the steady state response for $\hat{F}(z)$ was higher than the one for $F(s)$. Compute the static gains for the two systems, and use them to quantify the observed difference in (b).

*Solution*:

$$F(s=0) = \frac{1}{0+1} = 1 = 0dB$$

For $s=0$, we have $z = e^{sT} = e^{0T} = 1$. Hence, $\hat{F}(z=1) = \frac{T}{1-\alpha} = \frac{0.0314}{1-0.9051} = 1.0507 = 0.43dB$.

This is confirmed in the plot.

**PROBLEM 4(25pts)** Consider the continuous-time transfer function: $G_p(s) = \dfrac{2}{s^2+s+9} \overset{\Delta}{=} \dfrac{Y(s)}{U(s)}$.

**(a)(9pts)** To obtain a discrete-time approximation, call it $G_p(z)$, (i) find $a$ and $b$ such that $G_p(s) = c\left(\dfrac{b}{(s+a)^2+b^2}\right)$. (ii) For $T = 0.03142\,\text{sec}$ use the transform pair 21 in Table 8.1 on p.620 to compute $G_p(z)$. (iii) Use the c2d(Gp,T,'impulse') command to obtain $G_{imp}(z)$. (iv) Show that the result from Table 1 is off by a factor of $T$ from the result in (iii).

<u>*Solution*</u>: [See code @ 4(a).]

(i): $G_p(s) = \dfrac{2}{s^2+s+9} = \left(\dfrac{2}{\sqrt{8.75}}\right)\dfrac{\sqrt{8.75}}{(s+0.5)^2+s+8.75}$. (ii): $\dfrac{b}{(s+a)^2+b^2} \leftrightarrow \dfrac{ze^{-aT}\sin(bT)}{z^2-2ze^{-aT}\cos(bT)+e^{-2aT}}$

b1 = 0.0914 a1 = -1.9603 a0 = 0.9691. Hence: $G_p(z) = \left(\dfrac{2}{b}\right)\dfrac{b_1 z}{z^2+a_1 z+a_0} = \dfrac{0.06177z}{z^2-1.9603z+.9691}$.

(iii): Gimp = 0.01941 z / ( z^2 - 1.96 z + 0.9691) (iv): 0.06177*T= 0.019. Hence, the table does NOT include $T$, *while Matlab's* c2d command with the 'imp' flag does.

**(b)(10pts)** For T=0.03142 sec ($\omega_s = 200r/s$) using 'c2d(Gp,T,'zoh')' gives: $G_{zoh}(z) = \dfrac{.000976z+.000966}{z^2-1.96z+.9691}$. By including the 'zoh' flag, Matlab includes a zero-order-hold transfer function: $G_{zoh}(s) = (1-z^{-1})/s = (z-1)/sz$. Consequently, $G_{zoh}(z) = Z[G_{zoh}(s)G_p(s)] = \left(\dfrac{z-1}{z}\right)Z\left\{\dfrac{G_p(s)}{s}\right\}$. Use entry 22 in Table 8.1 to show that, indeed, this is the case.

<u>*Solution*</u>: $Z\left\{\dfrac{G_p(s)}{s}\right\} = Z\left\{\dfrac{2}{s[(s+a)^2+b^2]}\right\} = \left(\dfrac{2}{a^2+b^2}\right)Z\left\{\dfrac{a^2+b^2}{s[(s+a)^2+b^2]}\right\} = \left(\dfrac{2}{a^2+b^2}\right)\left(\dfrac{z(Az+B)}{(z-1)(z^2-2e^{-aT}\cos(bT)z+e^{-2aT})}\right)$

Hence: $G_p(z) = \left(\dfrac{2}{a^2+b^2}\right)\left(\dfrac{Az+B}{z^2-2e^{-aT}\cos(bT)z+e^{-2aT}}\right)$, where $A = 1-e^{-aT}\cos(bT)-\left(\dfrac{a}{b}\right)e^{-aT}\sin(bT) = \mathbf{0.0044}$ &

$B = e^{-2aT}+\left(\dfrac{a}{b}\right)e^{-aT}\sin(bT)-e^{-aT}\cos(bT) = \mathbf{0.0043}$. $\dfrac{2}{a^2+b^2} = \dfrac{2}{9} = \mathbf{0.222}$ Hence: $G_p(z) = \dfrac{.000976z+.000966}{z^2-1.96z+.9691}$.

**(c)(6pts)** From (b) we see that, even though the 'c2d' command gives $G_p(z)$, this is <u>not</u>** a transfer function in discrete time. Rather, it is in continuous time due to the inclusion of the ZOH digital-to-analog (D/A) circuit. Overlay plots of the step responses for $G_p(s)$, $G_{imp}(z)$ and $G_{zoh}(z)$ to see this visually.

**This is the case if ZOH is a circuit. If ZOH is an approximate integration method then $G_p(z)$ <u>is</u> in discrete time. In other words, the use of 'zoh' is ambiguous. It can be used for numerical integration, or it can represent a circuit.

<u>*Solution*</u>: [See code @ 3(c).]

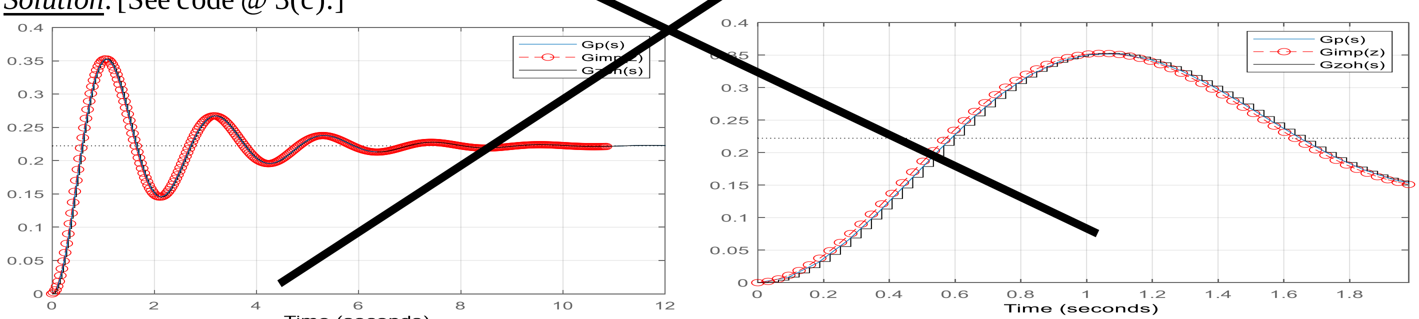


**Figure 4(c)** Step responses (LEFT) and zoomed (RIGHT) for $G_p(s)$, $G_{imp}(z)$ and $G_{zoh}(s)$.

## Appendix Matlab Code

```
%PROGRAM NAME: hw5.m   (4/3/20)
%======================================
%PROBLEM 1
%(a): G(s)=1/(s^2+0.2s+100)
g=@(w)abs((1i*w).^2+0.2*(1i*w)+100).^-2;
gint=(1/pi)*integral(g,0,inf)
%(b):
c=6.3;
VAR=25;
w=logspace(0,2,5000);
S=VAR*c^2*((100-w.^2).^2 +(0.2*w).^2).^-1;
SdB=10*log10(S);
figure(20)
semilogx(w,SdB)
title('Turbulence PSD')
xlabel('Frequency (rad/sec)')
ylabel('dB')
grid
%(c):
wN=1000;
del=pi/wN;
tmax=200;
npts=fix(tmax/del);
STD=sqrt(VAR);
G=tf(STD*c,[1 0.2 100]);
u=normrnd(0,1/sqrt(del),1,npts);
t=0:del:(npts-1)*del;
y=lsim(G,u,t);
figure(21)
plot(t,y)
title('Simulated Turbulence')
xlabel('Time (sec.)')
grid
%=================================
%PROBLEM 2
%(b):
F=tf(1,[1 1]);
figure(20)
bode(F)
grid
%(g):
wN=31.5; T=pi/wN; A=exp(-T);
w=0.01:0.01:wN;
M2=(1+w.^2).^-1;
M2dB=10*log10(M2);
TH=-atand(w);
Mhat2=T^2*(1+A^2 - 2*A*cos(w*T)).^-1;
MM2dB=10*log10(Mhat2);
THhat=-atand(sin(w*T)./(1-A*cos(w*T)));
figure(21)
semilogx(w,[M2dB;MM2dB])
title('FRF Magnitudes')
legend('M(w)','Mhat(w)')
xlabel('Frequency (rad/sec)')
ylabel('dB')
grid
figure(22)
semilogx(w,[TH;THhat])
title('FRF Phases')
legend('TH(w)','THhat(w)')
grid
xlabel('Frequency (rad/sec)')
ylabel('Degrees')
%==================================
%PROBLEM 3
%(a):
figure(30)
bode(F)
hold on
Fhat=tf([T 0],[1 -A],T);
bode(Fhat)
legend('F','Fhat')
```

```
grid
%(b):
figure(31)
step(F)
hold on
step(Fhat)
legend('y(t)','yhat(t)')
title('Unit Step Responses')
grid
%===================================
%PROBLEM 4
Gp=tf(2,[1 1 9]);
z=tf('z',T);
%(a):
%(ii):
T=0.03142;
a=0.5; %zeta*wn
b=sqrt(8.75); %wd
b1=exp(-a*T)*sin(b*T);
a1=-2*exp(-a*T)*cos(b*T);
a0=exp(-2*a*T);
c=2/b;
Ga=c*b1/(z^2+a1*z+a0);
Gimp=c2d(Gp,T,'imp');
%(b):
B=exp(-2*a*T)+exp(-a*T)*((a/b)*sin(b*T)-cos(b*T));
A=1-exp(-a*T)*(cos(b*T)+(a/b)*sin(b*T));
c=2/(a^2+b^2);
a1=-2*exp(-a*T)*cos(b*T);
a0=exp(-2*a*T);
G_theory=c*(A*z+B)/(z^2+a1*z+a0)
Gzoh=c2d(Gp,T,'zoh')
%(c):
figure(30)
step(Gp)
hold on
[gimp,t]=step(Gimp,'red');
plot(t,gimp,'ro--')
step(Gzoh,'k')
legend('Gp(s)','Gimp(z)','Gzoh(s)')
grid
```