Homework 2 Spring 2020 DUE 2/7/20 SOLUTION

PROBLEM 1. (15 pts) Consider the system described by: $\ddot{y}+0.2\dot{y}+100y=10x$

(a)(3 pts) Give the system *transfer function***:** $G_p(s)=10/(s^2+0.2s+100)$

(b)(2 pts) Give the system *static gain***:** $g_s=G_p(0)=0.1$

(c)(2 pts) Give the system *undamped natural frequency***:** $\omega_n=\sqrt{100}=10$

(d)(3 pts) Compute the system *damping ratio***:** ξ : $2\xi\omega_n=0.2\Rightarrow\xi\omega_n=0.1\Rightarrow\xi=0.1/10\Rightarrow\underline{\xi=0.01}$

(e)(2 pts) Compute the system *damped natural frequency***:** $\omega_d=\omega_n\sqrt{1-\xi^2}\quad=\underline{10\sqrt{0.9999}}\cong 10$

(e)(3 pts) The two *poles* of $G_p(s)$ in (a) are $s_{1,2}=-\xi\omega_n\pm i\omega_d$. Express these in *polar coordinates*:

$$s_1=\rho e^{i\phi} where\ \underline{\rho=\sqrt{(-\xi\omega_n)^2+\omega_d^2}=\omega_n}\ \&\ \underline{\phi=\pi-\tan^{-1}(\sqrt{1-\xi^2}}/\xi=\underline{\pi-\cos^{-1}(\xi)}\quad ;\ s_2=\rho e^{-i\phi}$$

PROBLEM 2(15 pts)

(a)(8pts) A plot of a car's speed, in response to an accelerator pedal displacement is shown at right. Arrive at a differential equation that will model this behavior.

Figure 2(a) Pedal position & speed plots.

Solution: Since the response appears to be exponential, I assume a 1st order O.D.E:
$5\tau\cong 20\sec\Rightarrow\tau\cong\mathbf{4sec}$; $v_{ss}/\theta_{ss}=40/20\Rightarrow g_s=\mathbf{2}\,[mph/\text{deg.}]$.

Hence , my model is: $4\dot{v}+v=2\theta(t)$

(b)(7pts) The initial condition response of a satellite solar panel is shown at right. Use the log decrement method to estimate the damping ratio ζ .

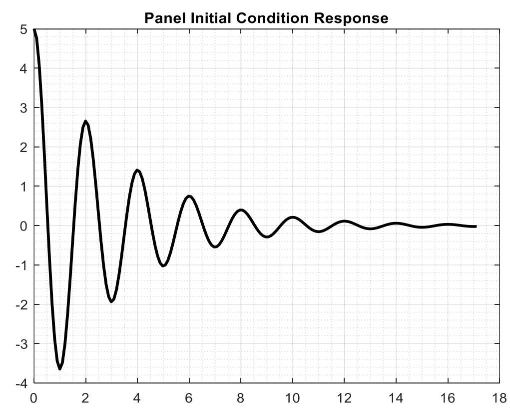


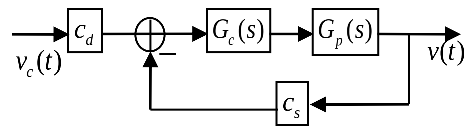
Figure 2(b) Panel initial condition response.

Solution:

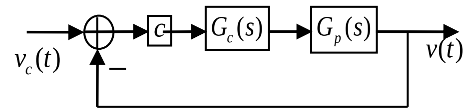
$y(2)\cong 2.6=ce^{-\zeta\omega_n 2}$ and $y(8)\cong 0.4=ce^{-\zeta\omega_n(2+3T)}$. So $\dfrac{y(2)}{y(8)}=6.5=\dfrac{ce^{-\zeta\omega_n 2}}{ce^{-\zeta\omega_n(2+3T)}}e^{3\zeta\omega_n T}$

where $\zeta\omega_n T=\dfrac{2\pi\zeta\omega_n}{\omega_d}=\dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$. So $\ln(6.5)=\dfrac{6\pi\zeta}{\sqrt{1-\zeta^2}}$. Squaring and solving for ζ gives: $\zeta=\dfrac{\ln(6.5)}{\sqrt{\ln(6.5)^2+(6\pi)^2}}=0.099\cong\mathbf{0.1}$

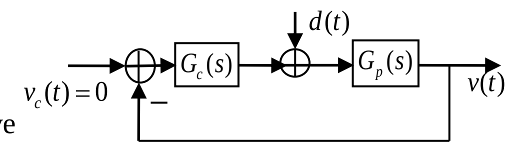
PROBLEM 3(15pts) The thrust/speed transfer function of a certain land vehicle is $G_p(s) = \dfrac{5}{s+0.25}\left[mph/lb_f\right]$.The vehicle speed *sensor* has a transfer function $G_s(s) = c_s\left[mv/mph\right]$. The command speed *dial*, which is a rotary potentiometer, has a transfer function $G_d(s) = c_d\left[mv/mph\right]$. The feedback control block diagram is shown at right.



(a)(5pts) Ideally, we have $c_s = c_d = c$. In this case, arrive at block diagram of a unity feedback control system that is mathematically the same as the above block diagram.

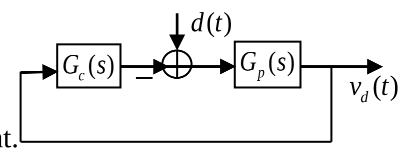


Solution: The summing junction output that goes into the controller is $c[v_c(t) - v(t)]$. Hence, we can frame the above diagram as shown at right.



(b)(5pts) The thrust from the controller to the plant will be countered by gravity-related forces such as hills. This can be modeled by the block diagram at right. In this part we will assume $c_s = c_d = c$. As you should have shown in (a), this constant can be mathematically lumped together with $G_c(s)$ Show that the speed/disturbance transfer function is: $W_d(s) \overset{\Delta}{=} \dfrac{V_d(s)}{D(s)} = \dfrac{G_p(s)}{1+G_c(s)G_p(s)}$. [Note: As shown in the figure, this assumes that $v_c(t) = 0$.]

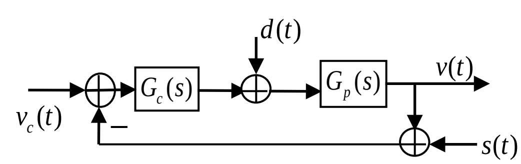
Solution:



Since $v_c(t) = 0$, we can replace the command summing junction by a block with a gain of -1. We can then transfer this sign to be a subtraction sign for the controller output into the disturbance summing junction. This results in the block diagram at right.

The forward loop transfer function is $G_p(s)$ and the feedback loop transfer function $G_c(s)$. Hence, the input to $G_p(s)$ is $D(s) - G_c(s)V_d(s)$. Hence the output from $G_p(s)$ is $V_d(s) = G_p(s)[D(s) - G_c(s)V_d(s)]$. This results in $\dfrac{V_d(s)}{D(s)} = \dfrac{G_p(s)}{1+G_c(s)G_p(s)}$.

(c)(5pts) Consider the block diagram at right that includes both disturbance and sensor noise. Using the same reasoning as should have been used in (b), it follows that the speed/sensor noise transfer function is:



$\dfrac{V_d(s)}{D(s)} = \dfrac{1}{1+G_c(s)G_p(s)}$. Suppose that the sensor noise is random and has a specified standard deviation $\sigma_s = \gamma[mv]$. Explain why this is not the standard deviation of $s(t)$ shown in the figure. Then determine what the standard deviation of that $s(t)$ is.

Explanation:

The units of $s(t)$ in the picture must be in *mph*. It makes no sense to add *mph* and *mv*. One needs to convert the units of $\sigma_s = \gamma[mv]$ to units of mph via $\sigma_{s_{mph}} = \gamma / c$.

PROBLEM 4(35pts) The position/torque transfer function for a robotic arm is $\frac{Y(s)}{T(s)} \stackrel{\Delta}{=} G_p(s) = \frac{10}{s^2 + 0.2s + 100}$. It is desired to design a unity feedback control system using a PD controller $G_c(s) = K_p + K_d s$.

(a)(3pts) Give the *closed loop* system transfer function).
<u>Answer</u>:

$$\frac{Y(s)}{Y_r(s)} \stackrel{\Delta}{=} W(s) = \frac{G_c G_p}{1 + G_c G_p} = \frac{10(K_d s + K_p)}{s^2 + (0.2 + 10K_d)s + (100 + 10K_p)}$$

(b)(5pts) Find the value of K_p so that the closed loop static gain is 0.95.
<u>Solution</u>:

$$W(0) = 0.95 = \frac{10K_p}{100 + 10K_p} \Rightarrow 95 + 9.5K_p = 10K_p \Rightarrow \mathbf{K_p = 190}.$$

(c)(8pts) For the value of K_p you found in (b), find the value of K_d such that the poles of the closed loop system are real and repeated (i.e. the system is critically damped).
<u>Solution</u>:

From (a) we have: $\omega_n^2 = 100 + 10(190) = 2000 \Rightarrow \omega_n = \mathbf{44.72}$. We also have: $2\xi\omega_n = 0.2 + 10K_d$. Since repeated real roots requires that $\xi = 1$, we obtain $2(44.72) = 0.2 + 10K_d$. Hence: $\mathbf{K_d = 8.924}$.

(d)(10pts) For the controller gains you obtained in (b) and (c), (i) use *Matlab* to compute and then plot the closed loop unit-step response. Then (ii) use this plot to validate the design static gain. Finally, (iii) discuss consequences of the critical damping specification on the response dynamics.
<u>Solution</u>: [See code @ 4(d).]

(i) W =(89.24 s + 1900)/(s^2 + 89.44 s + 2000). See BLUE @ right.

(ii) The dotted line shows that the static gain is 0.95.

(iii) The response does not oscillate, but it has notable overshoot.

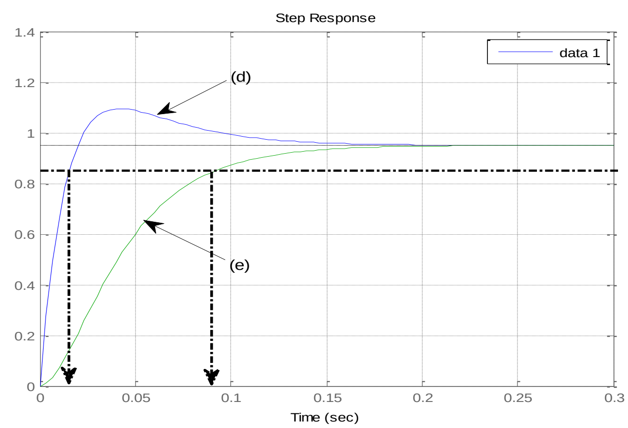


Figure 4(d & e) Unit step responses.

(e)(6pts) Remove the *s*-term from the numerator of your closed loop transfer function, and (i) overlay a plot of the step response for this system on the one in (d). Then (ii) discuss how it influences the response.
<u>Solution</u>: [See code @ 4(e).]

(i)For $W'(s) = \frac{1900}{s^2 + 89.44s + 2000}$ see above plot. (ii) The closed loop zero was responsible for the overshoot. This is because it is equivalent to the derivative of the input. In the case of a step, this is an impulse.

(f)(3pts) While overshoot is not desirable, you should have observed that it does have an advantage, in that the time it takes the response to achieved 90% of its steady state value is much smaller than if the closed loop zero were absent. From your two plots estimate how much faster it achieves this.

<u>Solution</u>: 90% of 0.95 is 0.855. The times at which the response cross this line are approximately .015 an .09. Hence, the system *zero* allows this level is achieved <u>~6 times faster</u>.

PROBLEM 5(20pts)

(a)(6pts) In 1(e) of Homework 1, you wrote a code to compute the roots of $p(s)=s^3+2s^2+25s+K$ as a function of K. Now consider the following equalities: $s^3+2s^2+25s+K=0 \;\Leftrightarrow\; 1+\dfrac{K}{s^3+2s^2+25s}=0$. So, define $G(s)\overset{\Delta}{=}\dfrac{1}{s^3+2s^2+25s}$. Then the roots of $p(s)=s^3+2s^2+25s+K$ are identical to the values of s that satisfy $1+KG(s)=0$. Even though we do not have any transfer functions, one can view this $KG(s)$ as an open loop transfer function associated with a closed loop system.
With this view, the values of s than satisfy $1+KG(s)=0$ are the closed loop system poles. (i) Use the command 'rlocus' to obtain a plot of these poles as a function of K. Then (ii) discuss how this plot compares to the plot obtained from your code in 1(e).
Solution: [See code @ 5(a).] (ii) The plot is the same as in 1(e).

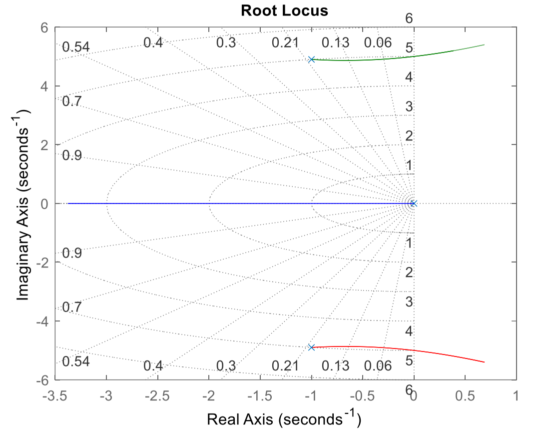
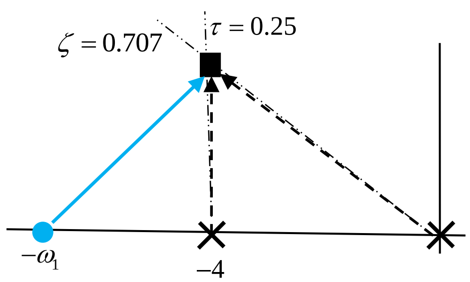


Figure 5(a) Root locus for K=0:0.1:100.

(b)(15pts) A certain DC motor position/voltage transfer function $G_p(s)=\dfrac{10}{s(s+4)}$. You are to use a forward loop PD controller $G_c(s)=K(s+\omega_1)$ so that the closed loop system has optimal damping (i.e. $\zeta=0.707$) and a time constant $\tau=0.25\,\text{sec}$

(i)(5pts) Obtain the value of ω_1 using the root locus angle criterion.
Provide a sketch to show how you arrive at the answer.
Solution:

$$\varphi_z-(135^o+90^o)=-180^o \Rightarrow \varphi_z=45^o \Rightarrow \omega_1=\mathbf{8}$$



(ii)(5pts) Use the root locus magnitude criterion to find K.
Solution:

$$1=\left|\frac{10K(s+4)}{s(s+4)}\right|=\frac{10K4\sqrt{2}}{(4\sqrt{2})(4)}=2.5K \quad \Rightarrow \mathbf{K=0.4}$$

(iii)(5pts) Plot the resulting closed loop step response. Then use it to Validate your design. [See code @ 5(b).]
Solution: [See code @ 5(b).]

The response has little overshoot and achieves steady stat in ~1.2 sec. This validates the design.

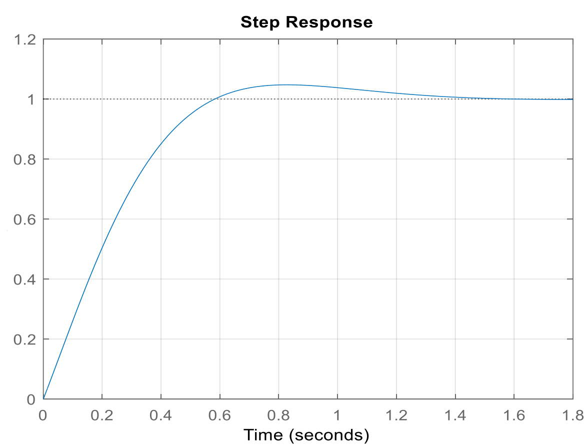


Figure 5(b) Closed loop step response.

Appendix Matlab Code

```
%PROGRAM NAME: hw2.m
%PROBLEM 2:
%(a): IN SOLUTION ONLY
tau=4; gs=2;
tho=20;
Ga=tf(gs,[tau 1]);
[v,t]=step(tho*Ga);
nt=length(t);
th=tho*ones(nt,1);
%-----------------
fig = figure(200);
left_color = [0 0 0];
right_color = [0 0 0];
set(fig,'defaultAxesColorOrder',[left_color; right_color]);
figure(200)
yyaxis left
plot(t,v,'k','LineWidth',2)
ylabel('mph','Color','k')
yyaxis right
plot(t,th,'k--','LineWidth',2)
ylabel('degrees','Color','k')
grid
title('Plot of Pedal Position and Resulting Speed')
xlabel('Time (sec)')
%-----------------
%(b):
Td=2; %Damped natural period
wd=2*pi/Td;
z=0.1; %Damping ratio
wn=wd/sqrt(1-z^2);
%For initial condition yo, compute Gb numerator:
yo=5;
Gb=tf([1 2*z*wn]*yo,[1 2*z*wn wn^2]);
[y,t]=impulse(Gb);
figure(201)
plot(t,y,'k','LineWidth',2)
title('Panel Initial Condition Response')
grid on
set(gca, 'yminorgrid', 'on')
set(gca, 'xminorgrid', 'on')
%set(gca,'MinorGridLineStyle','..')
set(gca,'GridLineStyle','-')
%============================
%PROBLEM 4:
%(d):
Gp=tf(10,[1 .2 100]);
Kp=190; Kd=8.924;
Gc=tf([Kd Kp],1);
G=Gc*Gp;
W=feedback(G,1);
figure(40)
step(W)
%(e):
[n,d]=tfdata(W,'v');
ne=n(3);
We=tf(ne,d);
hold on
step(We)
grid
%============================
%PROBLEM 5:
%(a):
G=tf(1,[1 2 25 0]);
K=0:.1:100;
figure(50)
rlocus(G,K)
```

```
%---------
%(b):
Gp=tf(10,[1 4 0]);
s=tf('s');
K=.253; w1=8;
Gc=K*(s+w1);
W=feedback(Gc*Gp,1);
figure(51)
step(W)
```