Homework 1 AERE331 Spring 2020 Due 1/24(F) SOLUTION
Note: The solution to each part of a given problem (including all figures) must be placed directly beneath that part. If it is placed elsewhere it will be ignored. Unless stated otherwise, place all Matlab code in the Appendix.

PROBLEM 1 (25pts) The mathematics in this course centers on two topics: (i) differential equations, and (ii) polynomials. The goal of this problem is to get you to understand their connection. Suppose that a function of time $x(t)$ is differentiable [i.e. $dx/dt \overset{\Delta}{=} \dot{x}(t)$ exists], and that it has the initial condition $x(t=0_-) \overset{\Delta}{=} x_0$. The *Laplace transform* of $x(t)$ is defined as: $\ell(x)(s) = \int_{t=0_-}^{\infty} x(t)e^{-st}dt \overset{\Delta}{=} X(s)$, where $s = \sigma + i\omega$ is allowed to be a complex number.

(a)(5pts) Recall from integral calculus: *integration by parts*: $\int u\,dv = u\,v - \int v\,du$. Use this to show that, so long as $\lim_{t\to\infty} x(t)e^{-st} = 0$, we have the following Laplace transform relation for $\dot{x}(t)$: $\ell(\dot{x}) = \int_{t=0_-}^{\infty} \dot{x}(t)e^{-st}dt = s\,X(s) - x_0$

Solution: Let $u = x(t)$ and $dv = e^{-st}dt$. Then $v = -\frac{1}{s}e^{-st}$ and $du = \dot{x}(t)\,dt$. It follows that

$\int_0^{\infty} x(t)e^{-st}dt = -\frac{x(t)}{s}e^{-st}\Big|_{t=0}^{\infty} + \frac{1}{s}\int_0^{\infty}\dot{x}(t)\,e^{-st}dt = \frac{x_0}{s} + \frac{1}{s}\int_0^{\infty}\dot{x}(t)\,e^{-st}dt$. Rearranging this gives: $\ell(\dot{x}) = s\,X(s) - x_0$.

(b)(5pts) Define $g(t) \overset{\Delta}{=} \dot{x}(t)$ with initial condition $g(0_-) = g_0 = \dot{x}_0$. Use (a) to show that $\ell(\ddot{x}) = s^2X(s) - sx_0 - \dot{x}_0$.
Solution: From (a) we have $\ell(\dot{g}) = sG(s) - g_0$, which is exactly: $\ell(\ddot{x}) = s\,\ell(\dot{x}) - \dot{x}_0$. Substituting the result in (a) gives $\ell(\ddot{x}) = s[sX(s) - x_0] - \dot{x}_0 = s^2X(s) - sx_0 - \dot{x}_0$.

(c)(5pts) From (a-b) it should be clear that the Laplace transform of $x^{(n)}(t)$ includes the term $s^nX(s)$ plus other terms that include the initial conditions. To see how polynomials enter into the picture, consider the second order differential equation: $\ddot{x} + 2\dot{x} + 25x = 10\,f(t)$ with initial conditions $\dot{x}_0$ and x_0. Take the Laplace transform of this equation, and solve it for the variable $X(s)$. Express $X(s)$ as the sum of TWO parts: one that depends on $F(s)$ and one that depends on $(\dot{x}_0, x_0)$.
Solution: $\ell(\ddot{x}) + 2L(\dot{x}) + 25L(x) = 10\ell(f) \;\Rightarrow\; [s^2X(s) - sx_0 - \dot{x}_0] + 2[sX(s) - x_0] + 25X(s) = 10F(s)$. Gathering terms gives: $(s^2 + 2s + 25)X(s) = F(s) + [2(s+1)x_0 + \dot{x}_0]$. Hence: $X(s) = \left(\frac{10}{s^2 + 2s + 25}\right)F(s) + \frac{(s+2)x_0 + \dot{x}_0}{s^2 + 2s + 25}$.

(d)(5pts) Your answer in (c) should involve the polynomial $p(s) \overset{\Delta}{=} s^2 + 2s + 25$. This is called the system *characteristic polynomial*. In fact, $p(s)$ should be present in the denominator of each of the terms. Hence, when $s = \sigma + i\omega$ is a root of $p(s)$, these terms 'explode'. Use the *Matlab* command 'roots' to obtain the roots of $p(s)$. [Include your code HERE.]
Solution: >> p=[1 2 25]; >> rp=roots(p) rp = -1.0000 +/- 4.8990i

(e)(5pts) Part (d) involved a polynomial, but in an almost trivial way since it was a quadratic. Consider the polynomial $p(s) = s^3 + 2s^2 + 25s + K$. Write a *Matlab* code that will plot the roots (use * not lines!) of $p(s)$ for *K*=0:0.1:100. Then use the data cursor to find the values of the purely imaginary roots when they hit the imaginary axis. Finally, substitute one of those purely imaginary values into $p(s)$, and solve it for the corresponding *K* value.
Solution: [See code @ 1(e).]

$$p(s = i5) = (i5)^3 + 2(i5)^2 + 25(i5) + K = 0 \Rightarrow \quad K = 50$$

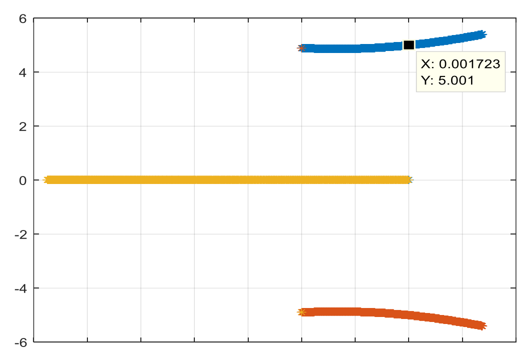
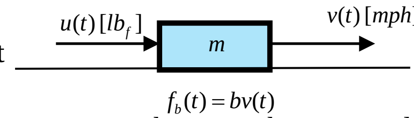


Figure 1(e) Roots of $p(s)$ as a function of *K*.

PROBLEM 2 (25pts) The goal of this problem is to give you an appreciation for the value of using Laplace transforms to solve O.D.E.s. This will be couched in the context of the figure at right. A force *input* $u(t)$ is applied to a mass, *m*, causing its velocity (the *output*) to move with a velocity $v(t)$. The only retarding force is viscous friction, $f_b(t)=bv(t)$, between the mass and the surface. A force balance gives the O.D.E.: $m\dot{v}+bv=u$, with initial condition v_0.



(a)(5pts) Show that $V(s)=\frac{v_o\tau}{\tau s+1}+\left[\frac{g_s}{\tau s+1}\right]U(s)$, where we have defined $\tau\overset{\Delta}{=}m/b$ and $g_s\overset{\Delta}{=}1/b$

Solution: $\ell[m\dot{v}+bv=u]\Rightarrow m[sV(s)-v_0]+bV(s)=U(s)\Rightarrow(\tau s+1)V(s)=\tau v_0+g_sU(s)$. The result follows.

(b)(5pts) From (a), we see that $V(s)$ [hence $v(t)$] is composed of two terms. Let $V_1(s)\overset{\Delta}{=}\frac{v_o\tau}{\tau s+1}$. Identify the appropriate entry in the Table of Laplace Transforms inside the front cover of the book. Then use it to show that $v_1(t)=v_0e^{-t/\tau}$.

Solution: Entry #7 is: $\frac{1}{s+a}\leftrightarrow e^{-at}$. Hence $V_1(s)\overset{\Delta}{=}\frac{v_o\tau}{\tau s+1}=\frac{v_o}{s+1/\tau}\leftrightarrow\quad v_1(t)=v_oe^{-t/\tau}$.

(c)(5pts) Let $V_2(s)\overset{\Delta}{=}\frac{g_s}{\tau s+1}\frac{u_0}{s}$ where we have assumed the force input is a step $u(t)=u_0 1(t)$ with Laplace transform $U(s)=u_0/s$. Give the appropriate table entry # and expression. Then use it to show that $v_2(t)=g_su_0(1-e^{-t/\tau})$.

Solution: Entry #11 is: $\frac{a}{s(s+a)}\leftrightarrow 1-e^{-at}$. Hence, $V_2(s)\overset{\Delta}{=}\frac{g_s}{\tau s+1}\frac{u_0}{s}=\frac{(1/\tau)g_su_0}{s(s+1/\tau)}\quad\leftrightarrow\quad v_2(t)=g_su_0(1-e^{-t/\tau})$.

Remark 2(c): The system step response includes two parameters. The parameter $g_s\overset{\Delta}{=}1/b$ is called the system *static gain*. It is the ratio of the steady state (i.e. 'static') output divided by the steady state (i.e. 'static') input. The parameter $\tau\overset{\Delta}{=}m/b$ is called the system *time constant*. The response will be within ~2% of the steady state response at time $t=4\tau$ (since $e^{-4}=0.0183\cong 0.02$).

(d)(5pts) Since $V(s)=V_1(s)+V_2(s)$, from (b-c) we have $v(t)=v_1(t)+v_2(t)=v_0e^{-t/\tau}+g_su_0(1-e^{-t/\tau})$. In words, $v(t)$ is the superposition of the i.c. response and the force step response. In this part assume that $v_0=0$. Then the steady state mass velocity will be $v_{ss}=\lim_{t\to\infty}v(t)=g_su_0$. Suppose that the viscous damping coefficient is $b=10\,[lb_f/mph]$. Find the value of u_0 such that $v_{ss}=50\,mph$.

Solution: $v_{ss}=50\,mph=g_su_0=u_0/b\quad\Rightarrow\quad u_0=50(b)=50(10)=500\,lb_f$.

(e)(5pts) Find the maximum value of the *weight* (in units of pounds) associated with the mass, *m*, such that $|v_{ss}-v(t)|\le 0.5\,mph$ for all $t\ge 20\sec$. [Hint: Recall that the units of *b* are lb_f/mph & those of τ are *sec*.]

Solution: $v_{ss}-v(20)=v_{ss}-g_su_0(1-e^{-20/\tau})=g_su_0e^{-20/\tau}\Rightarrow e^{-20/\tau}=\frac{v_{ss}-v(20)}{g_su_0}=\frac{0.5}{50}=0.01$, so that $\tau=-20/\ln(0.01)=4.34\sec$.

Since $\tau\overset{\Delta}{=}m/b$, we have $m=\tau b=4.34\sec\times 10\,lb_f/mph=43.4\frac{lb_f-\sec-hr}{miles}\times\frac{3600\sec}{hr}\times\frac{1mile}{5280\,ft}=29.6\frac{lb_f}{ft/s^2}$ or

$w=mg=29.6\frac{lb_f}{ft/s^2}\times 32.18\,ft/s^2=952.5^{\#}$.

PROBLEM 3 (25pts) This course is all about *transfer functions*. For a system defined by the O.D.E. $b_n y^{(n)} + b_{n-1} y^{(n-1)} + \ldots + b_1 \dot{y} + b_0 y = a_m x^{(m)} + a_{m-1} x^{(m-1)} + \ldots + a_1 \dot{x} + a_0 x$, the system *transfer function* is the ratio $\frac{Y(s)}{X(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0}$. Notice that this is obtained by taking the Laplace transform of the O.D.E. **under *zero initial conditions***, and solving for the ratio $Y(s)/X(s)$. Let $Y(s)/X(s) \stackrel{\Delta}{=} G(s)$. Then for any specified input $X(s)$, the output is simply $Y(s) = G(s)X(s)$. To arrive at $y(t)$ one can simply use a table of Laplace transform pairs. In this problem we will consider the transfer function:

$$G(s) = \frac{X(s)}{F(s)} = \frac{10}{s^2 + 4s + 25}. \quad (1)$$

(a)(5pts) Suppose that the input to this system is chosen to be a unit *impulse*: $f(t) = \delta(t)$. It is then reasonable to refer to the resulting output $y(t)$ as the system *impulse response*. Show that for any arbitrary transfer function, $G(s)$, the impulse response is simply $g(t)$. [Hint: See the entry for an impulse in the Table of Laplace Transform pairs.]

<u>*Solution*</u>: Since $f(t) = \delta(t) \quad \leftrightarrow \quad F(s) = 1$, then $X(s) = G(s)F(s) = G(s)$. Hence, $x(t) = g(t)$.

(b)(5pts) Identify the most appropriate entry in the table of Laplace transforms, and then use it to arrive at the impulse response associated with (1). [Hint: The system poles are a complex conjugate pair.]

<u>*Solution*</u>: Write $p(s) = s^2 + 2s + 25$ as a completed square: $p(s) = s^2 + 4s + 4 + 21 = (s+2)^2 + 4.5826^2$. We can then use entry #20 to arrive at $g(t)$: $G(s) = \frac{10}{s^2 + 4s + 25} = \left(\frac{10}{4.5826}\right)\frac{4.5826}{(s+2)^2 + 4.5826^2} \quad \Rightarrow \quad g(t) = 2.18 e^{-2t} \sin(4.5826 t)$.

(c)(5pts) Overlay a plot of your expression in (b) against a plot obtained using the *Matlab* commands ‘tf’ and ‘impulse’. Comment on how well they compare.

<u>*Solution*</u>: [See code @ 3(c).]

<u>*Comment*</u>: They are identical.

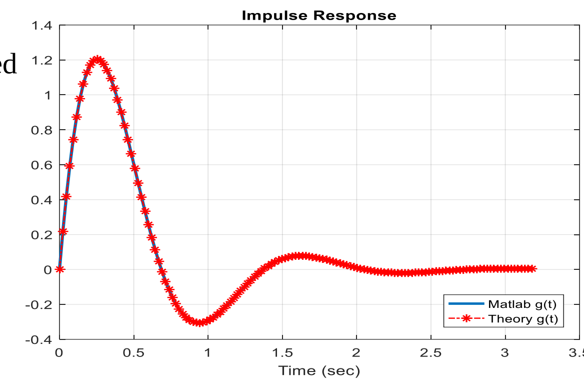


Figure 3(c) Matlab and theoretical impulse response.

(d)(5pts) Use the *Matlab* command ‘step’ to obtain a plot of the system response to a unit step. Then, from this plot use the data cursor to estimate the system static gain, and compare it to what it should be.

<u>*Solution*</u>: [See code @ 3(d).]

The true static gain is $G(s=0) = 10/25 = 0.4$. This compares well to that given in the plot.

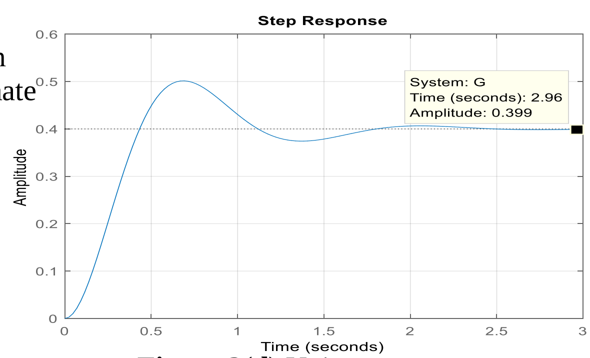
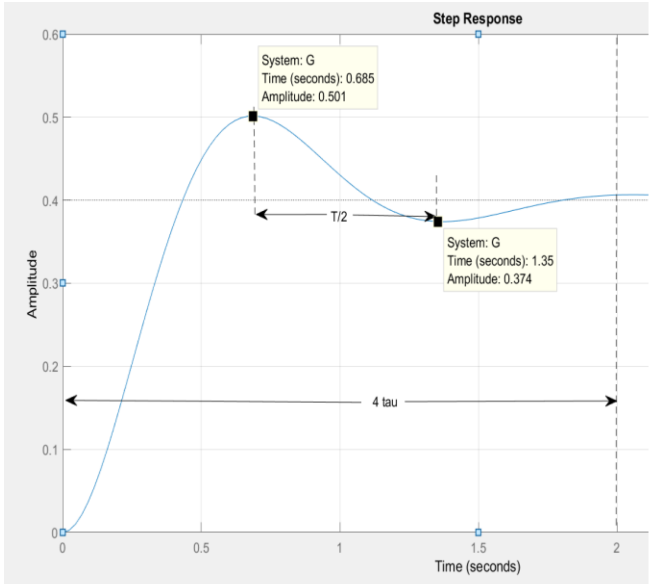


Figure 3(d) Unit step response.

(e)(5pts) Use your plot in 3(d) [or include a new one here] to estimate the system time constant and the damped natural frequency. Compare them to the values associated with your theoretical response in (b).



Solution:

$g(t) = 2.18e^{-2t}\sin(4.5826t) \quad \Rightarrow \quad \tau = 1/2\,\text{sec} \quad \& \quad \omega_d = 4.5826 r/s$

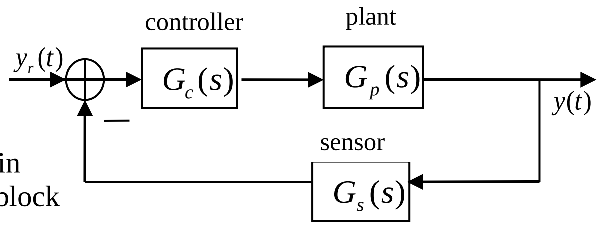
From the information in the plot we have:

$4\tau = 2\,\text{sec} \quad \Rightarrow \quad \tau = 1/2\,\text{sec}$, which is exactly what we should get.

$T_d/2 = 1.35 - 0.685 \quad \Rightarrow \quad T_d = 1.33\text{sec} \quad \Rightarrow \quad \omega_d = 2\pi/T_d = 4.72 r/s$.

This is slightly higher than the true value 4.5826 r/s.

PROBLEM 4 (25pts) Consider the feedback control system at right.



(a)(5pts) The beauty of transfer functions is that in the Laplace domain You simply multiply blocks. For example, the input to the controller block is the error $E(s)\stackrel{\Delta}{=}Y_r(s)-G_s(s)Y(s)$. When this input is run through the controller and the plant blocks, the output is simply $Y(s)=G_p(s)G_c(s)E(s)$. Use these relations to show that the closed loop transfer function is $W(s)\stackrel{\Delta}{=}\frac{Y(s)}{Y_r(s)}=\frac{G_p(s)G_c(s)}{1+G_p(s)G_c(s)G_s(s)}$.

<u>Solution</u>: $Y(s)=G_p(s)G_c(s)[Y_r(s)-G_s(s)Y(s)] \ \Rightarrow [1+G_p(s)G_c(s)G_s(s)]Y(s)=G_p(s)G_c(s)Y_r(s)$. The result follows immediately.

(b)(5pts) The closed loop system <u>poles</u> are the values of s that make $W(s)=\infty$, while the closed loop <u>zeros</u> are the values of s that make $W(s)=0$. It should be clear that the poles must satisfy $1+G_p(s)G_c(s)G_s(s)=0$. Suppose that we have the following specific transfer functions: $G_p(s)=\frac{1}{s(s^2+2s+25)}$, $G_c(s)=K(s+5)$, and $G_s(s)=1$. Find the expression for the polynomial whose roots are the closed loop poles.

<u>Solution</u>: $1+G_p(s)G_c(s)G_s(s)=0=1+\frac{K(s+5)}{s(s^2+2s+25)} \ \Rightarrow \ s(s^2+2s+25)+K(s+5)=0$.

Hence: $p(s)=s(s^2+2s+25)+K(s+5)=s^3+2s^3+(K+25)s+5K$.

(c)(5pts) The closed loop system will be stable if all the poles are in the proper Left Half Plane (LHP). Your polynomial in (b) should be a cubic polynomial, and should include the controller gain parameter *K*. You had a similar situation in PROBLEM 1(e), and so you could modify that code to find the range of *K* values such that the system will be stable. Instead of using such a code, use the *Matlab* command 'rlocus' to obtain a plot of the closed loop poles as a function of *K*. Then use the data cursor to identify the largest value of *K* such that the system will be stable. [Note: The rlocus <u>argument</u> is the *open loop transfer function* $G_p(s)G_c(s)G_s(s)$ with *K* set to 1.0. With this argument, the 'rlocus' code computes the *s*-values that satisfy $1+KG_p(s)G_c(s)G_s(s)=0$ for a range of *K*-values [i.e. it does exactly what your code in 1(e) did].

<u>Solution</u>: [See code @ 4(c).] The maximum *K* is 16.6.

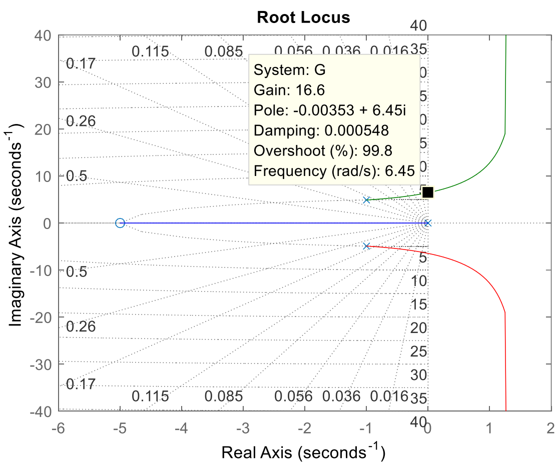
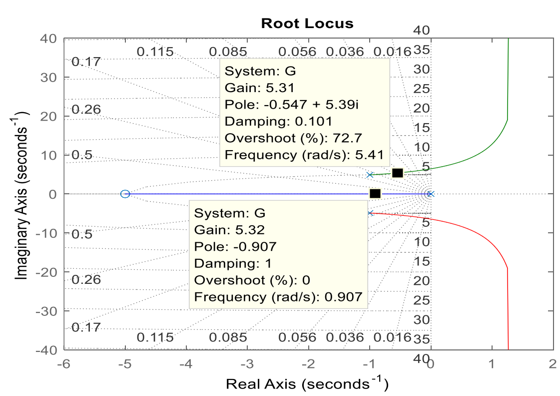


Figure 4(c) Closed loop root locus.

(d)(5pts) I used the data cursor to find the gain *K* (=5.31) that would result in complex-conjugate closed loop poles having a damping ratio $\zeta\cong 0.1$. The data cursor data also gives the corresponding pole values $s_{1,2}=-0.547\pm i5.39$. I then used the cursor to find the value of the third real pole $s_3=-0.907$ for this value of *K*. The conjugate poles have a time constant, $\tau_{s_{1,2}}$, and the real pole has a time constant, τ_{s_3}. Arrive at the values for these two time constants.



<u>Solution</u>: The time constant associated with a pole is equal to the negative of the inverse of the real part of the pole.

Hence: $\tau_{s_{1,2}}=1/0.547\cong 1.83\text{sec}$ and $\tau_{s_3}=1/0.907\cong 1.10\text{sec}$

Figure 4(d) Closed loop root locus with pole information.

(e)(5pts) In relation to the real pole, the data cursor information in Figure 4(d) states that $\zeta=1$. Explain why this is, at the very least, misleading. [Hint: Think of the meaning of the terms *underdamped, critically damped* and *overdamped*.]

<u>Explanation</u>: As ζ increases, a 2nd order system becomes *critically damped* when $\zeta=1$. Beyond that point it has two *real* roots. Hence, a damping ratio is no longer well-defined.

Appendix Matlab Code

```
%PROGRAM NAME: hw1.m
%PROBLEM 1(e):
K=0:.1:100;
n=length(K);
rp = zeros(n,3);
for k=1:n
    rp(k,:)=roots([1 2 25 K(k)]);
end
RE=real(rp);
IM=imag(rp);
figure(1)
plot(RE,IM,'*')
grid
%=======================================
%PROBLEM 3
%(c):
G=tf(10,[1 4 25]);
[gM,t]=impulse(G);
g=2.18*exp(-2*t).*sin(4.5826*t);
figure(30)
plot(t,gM,'LineWidth',2)
hold on
plot(t,g,'*-.r','LineWidth',1)
grid
xlabel('Time (sec)')
title('Impulse Response')
legend('Matlab g(t)','Theory g(t)','Location','SouthEast')
%(d-e):
figure(31)
step(G)
grid

%=======================================
%PROBLEM 4(c):
s=tf('s');
Gp=1/(s*(s^2+2*s+25));
Gc=s+5;
Gs=1;
G=Gc*Gp*Gs;
figure(40)
rlocus(G)
grid
```