Butterworth order-M LPF with cutoff frequency ω_c: $|H(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^{2M}} \Rightarrow H(s)H(-s) = \frac{1}{1+\left(\frac{s}{j\omega_c}\right)^{2M}}$

Normalized ($\omega_c = 1$) LPF **transformations**: $\begin{cases} lp2hp: \quad s \to \frac{\omega_c}{s} \\ lp2bp: s \to \frac{s^2+\omega_0^2}{Bs} \\ lp2br: s \to \frac{Bs}{s^2+\omega_0^2} \end{cases}$

Order-M FIR filter introduces $\frac{M}{2}$ delay for desired freq response:, $H_d\left(e^{j\Omega}\right) = H\left(e^{j\Omega}\right)e^{-j\Omega M/2} \leftrightarrow \boldsymbol{h_d}[\boldsymbol{n}]$.
Then, $\boldsymbol{h_{FIR}}[\boldsymbol{n}] = \boldsymbol{h_d}[\boldsymbol{n}].\boldsymbol{w_M}[\boldsymbol{n}]$, where $w_M[n]$ is order-M window to truncate beyond $M+1$ impulses.

IIR filter: $H(j\omega) \xrightarrow{\text{analog design}} H(s) \xrightarrow{\text{prewarp: } \omega_p = \frac{2}{T}\tan\left(\frac{\omega T}{2}\right)} H_p(s) \xrightarrow{\text{bilinear: } s=\frac{2}{T}\left(\frac{z-1}{z+1}\right) \Rightarrow \omega = \frac{2}{T}\tan^{-1}\left(\frac{\omega_p T}{2}\right)} H(z)$.
T chosen using Nyquist criterion; so if max freq of interest is $\omega_0 = 2\pi f_0$, then $T \le \frac{1}{2f_0} = \frac{1}{2\left(\frac{\omega_0}{2\pi}\right)} = \frac{\pi}{\omega_0}$.

Feedback system: forward-gain $G(s)$, open loop-gain $L(s)$ (= product of forward and feedback gains) ⇒
closed-loop gain, $T(s) = \frac{G(s)}{1+L(s)}$; **Model sensitivity,** $\frac{\frac{\Delta T}{T}}{\frac{\Delta G}{G}} = \frac{1}{1+L(s)}$; **Noise sensitivity,** $\frac{Y(s)}{N(s)}\Big|_{X(s)=0} = \frac{1}{1+L(s)}$.

Routh-Hurwitz: Analyze char. poly., $\chi(s) \coloneqq \mathrm{den}\big(T(s)\big) = \mathrm{num}(L(s)) + \mathrm{den}(L(s)) \overset{\text{def}}{=} \sum_{k=0}^{n} a_k s^k$;
s^n-row, k^{th} coeff: $\alpha_k^{(n)} \overset{\text{def}}{=} a_{n-2k}$; s^{n-1}-row, k^{th} coeff: $\alpha_k^{(n-1)} \overset{\text{def}}{=} a_{(n-1)-2k}$ $\left(0 \le k \le \left\lfloor\frac{n+1}{2}\right\rfloor\right)$;
s^j-row, k^{th} coeff: $\alpha_k^{(j)} \overset{\text{def}}{=} \frac{\alpha_0^{(j+1)}\alpha_{k+1}^{(j+2)} - \alpha_0^{(j+2)}\alpha_{k+1}^{(j+1)}}{\alpha_0^{(j+1)}}$ $\left(n-2 \ge j \ge 0,\ 0 \le k \le \left\lfloor\frac{j+1}{2}\right\rfloor\right)$;
#unstable poles = #sign-changes in 1st column of RH; row before a zero-row called **auxiliary polynomial** row; aux. poly. a factor of characteristic polynomial; 2nd degree aux. poly. ⇒ roots on imaginary-axis.

Root-locus: For loop-gain $L(s) = K\frac{N(s)}{D(s)}$, plot of roots of $\chi(s) \overset{\text{def}}{=} KN(s) + D(s)$, as $K \in [0, \infty]$.
#branches = max(deg(N), deg(D)); starts: poles of $L(s)$; ends: zeros of $L(s)$; symmetric wrt real-axis;
#asymptotes = |deg(N) – deg(D)|, angles = $\frac{\text{odd multiples of } \pi}{\text{\#asymptotes}}$, meeting-point = $\frac{\text{sum of poles} - \text{sum of zeros}}{\text{\#asymptotes}}$;
candidate breakaway points: $\left(\frac{d}{ds}\big(\chi(s)\big) = 0\right) \equiv \left(\frac{d}{ds}\big(L(s)\big) = 0\right) \equiv \left(\frac{d}{ds}\left(\frac{1}{L(s)}\right) = 0\right)$;
angle criterion: (sum of pole-vector angles) – (sum of zero-vector angles) = odd multiple of π;
magnitude criterion: gain K = (product of pole-vector lengths)/(product of zero-vector lengths);
Use RH on $\chi(s) \overset{\text{def}}{=} KN(s) + D(s)$ to find gain K at which roots are on imaginary-axis.

Nyquist-plot: Polar plot of loop-gain freq resp, $L(j\omega)$ for $\omega \in [-\infty, \infty]$ or of $L(e^{j\Omega})$ for $\Omega \in [\pi, -\pi]$;
#unstable poles of T = #unstable poles of L + #clockwise encirclements of $(-1,0)$ point by Nyquist-plot;
phase-margin = $\Big|\big|\angle L(j\omega_g)\big| - \pi\Big|$, when $|L(j\omega_g)| = 1$; gain-margin = $\frac{1}{|L(j\omega_p)|}$ dB, when $\angle L(j\omega_p) = \pi$.

z-Transforms

Signal	Transform	
$x[n] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1}\, dz$	$X[z] = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$[\cos(\Omega_1 n)]u[n]$	$\frac{1 - z^{-1}\cos\Omega_1}{1 - z^{-1}2\cos\Omega_1 + z^{-2}}$	$\lvert z\rvert > 1$
$[\sin(\Omega_1 n)]u[n]$	$\frac{z^{-1}\sin\Omega_1}{1 - z^{-1}2\cos\Omega_1 + z^{-2}}$	$\lvert z\rvert > 1$
$[r^n\cos(\Omega_1 n)]u[n]$	$\frac{1 - z^{-1}r\cos\Omega_1}{1 - z^{-1}2r\cos\Omega_1 + r^2z^{-2}}$	$\lvert z\rvert > r$
$[r^n\sin(\Omega_1 n)]u[n]$	$\frac{z^{-1}r\sin\Omega_1}{1 - z^{-1}2r\cos\Omega_1 + r^2z^{-2}}$	$\lvert z\rvert > r$

E.1.1 Bilateral Transforms for Signals that Are Nonzero for $n < 0$

Signal	Bilateral Transform	ROC
$u[-n-1]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$
$-\alpha^n u[-n-1]$	$\frac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
$-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$

z-Transform Properties

Signal	Unilateral Transform $x[n] \overset{z_u}{\longleftrightarrow} X(z)$, $y[n] \overset{z_u}{\longleftrightarrow} Y(z)$	Bilateral Transform $x[n] \overset{z}{\longleftrightarrow} X(z)$, $y[n] \overset{z}{\longleftrightarrow} Y(z)$	ROC $z \in R_x$, $z \in R_y$
$ax[n] + by[n]$	$aX(z) + bY(z)$	$aX(z) + bY(z)$	At least $R_x \cap R_y$
$x[n-k]$	See below	$z^{-k}X(z)$	R_x, except possibly $\lvert z\rvert = 0, \infty$
$\alpha^n x[n]$	$X\left(\frac{z}{\alpha}\right)$	$X\left(\frac{z}{\alpha}\right)$	$\lvert\alpha\rvert R_x$
$x[-n]$	—	$X\left(\frac{1}{z}\right)$	$\frac{1}{R_x}$
$x[n] * y[n]$	$X(z)Y(z)$ if $x[n] = y[n] = 0$ for $n < 0$	$X(z)Y(z)$	At least $R_x \cap R_y$
$nx[n]$	$-z\frac{d}{dz}X(z)$	$-z\frac{d}{dz}X(z)$	R_x, except possibly addition or deletion of $z = 0$

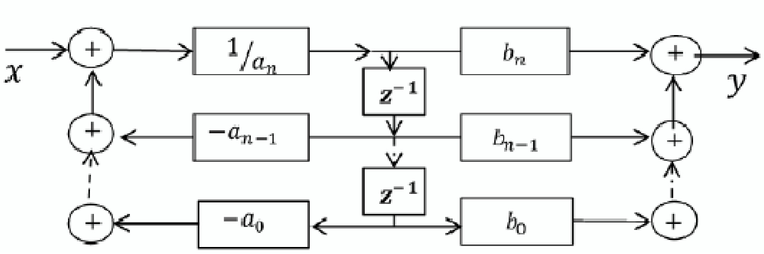
E.2.1 Unilateral z-Transform Time-Shift Property

$$x[n-k] \overset{z_u}{\longleftrightarrow} x[-k] + x[-k+1]z^{-1} + \cdots + x[-1]z^{-k+1} + z^{-k}X(z) \quad \text{for } k > 0$$

$$x[n+k] \overset{z_u}{\longleftrightarrow} -x[0]z^{k} - x[1]z^{k-1} - \cdots - x[k-1]z + z^{k}X(z) \quad \text{for } k > 0$$

Simulation Diagram:

$$\left[\sum_{j=o}^{n} a_j y[k+j] = \sum_{j=0}^{n} b_j x[k+j]\right] \Leftrightarrow H(z) = \frac{b_n z^n + \ldots + b_0}{a_n z^n + \ldots + a_0}.$$



Partial fraction corresponding to kth pole d_k having multiplicity r in expansion of $\frac{B'(z)}{A(z)}$: $\sum_{j=1}^{r}\frac{A_{kj}}{(1-d_k z^{-1})^j}$

Then, $\frac{A_{kj}}{(1-d_k z^{-1})^j} \leftrightarrow$

$$\begin{cases} A_{kj}\frac{(n+1)\ldots(n+j-1)}{(j-1)!}(d_k)^n u[n], & ROC: \lvert z\rvert > d_k \\ -A_{kj}\frac{(n+1)\ldots(n+j-1)}{(j-1)!}(d_k)^n u[-n-1], & ROC: \lvert z\rvert < d_k \end{cases}$$

where, $A_{kj} = \frac{1}{(r-j)!}\frac{d^{r-j}}{dz^{r-j}}\left[\frac{B'(z)}{A(z)}(1 - d_k z^{-1})^r\right]_{s=d_k}$.

Laplace Transforms

Signal $x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$	Transform $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$	ROC
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
$\delta(t-\tau), \quad \tau \ge 0$	$e^{-s\tau}$	for all s
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} > -a$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}\{s\} > -a$
$[\cos(\omega_1 t)]u(t)$	$\frac{s}{s^2+\omega_1^2}$	$\mathrm{Re}\{s\} > 0$
$[\sin(\omega_1 t)]u(t)$	$\frac{\omega_1}{s^2+\omega_1^2}$	$\mathrm{Re}\{s\} > 0$
$[e^{-at}\cos(\omega_1 t)]u(t)$	$\frac{s+a}{(s+a)^2+\omega_1^2}$	$\mathrm{Re}\{s\} > -a$
$[e^{-at}\sin(\omega_1 t)]u(t)$	$\frac{\omega_1}{(s+a)^2+\omega_1^2}$	$\mathrm{Re}\{s\} > -a$

■ D.1.1 Bilateral Laplace Transforms for Signals That Are Nonzero for $t < 0$

Signal	Bilateral Transform	ROC
$\delta(t-\tau), \tau < 0$	$e^{-s\tau}$	for all s
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} < 0$
$-tu(-t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} < 0$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} < -a$
$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}\{s\} < -a$

Signal	Unilateral Transform $x(t) \xleftrightarrow{\mathcal{L}_u} X(s)$, $y(t) \xleftrightarrow{\mathcal{L}_u} Y(s)$	Bilateral Transform $x(t) \xleftrightarrow{\mathcal{L}} X(s)$, $y(t) \xleftrightarrow{\mathcal{L}} Y(s)$	ROC $s \in R_x$, $s \in R_y$
$ax(t) + by(t)$	$aX(s) + bY(s)$	$aX(s) + bY(s)$	At least $R_x \cap R_y$
$x(t-\tau)$	$e^{-s\tau}X(s)$ if $x(t-\tau)u(t) = x(t-\tau)u(t-\tau)$	$e^{-s\tau}X(s)$	R_x
$e^{s_o t}x(t)$	$X(s - s_o)$	$X(s - s_o)$	$R_x + \mathrm{Re}\{s_o\}$
$x(at)$	$\frac{1}{a}X\left(\frac{s}{a}\right), \quad a > 0$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	$\frac{R_x}{\lvert a\rvert}$
$x(t) * y(t)$	$X(s)Y(s)$ if $x(t) = y(t) = 0$ for $t < 0$	$X(s)Y(s)$	At least $R_x \cap R_y$
$-tx(t)$	$\frac{d}{ds}X(s)$	$\frac{d}{ds}X(s)$	R_x
$\frac{d}{dt}x(t)$	$sX(s) - x(0^-)$	$sX(s)$	At least R_x
$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}\int_{-\infty}^{0^-} x(\tau)\,d\tau + \frac{X(s)}{s}$	$\frac{X(s)}{s}$	At least $R_x \cap \{\mathrm{Re}\{s\} > 0\}$

Partial fraction corresponding to kth pole d_k having multiplicity r in expansion of $\frac{B'(s)}{A(s)}$: $\sum_{j=1}^{r}\frac{A_{kj}}{(s-d_k)^j}$

Then, $\frac{A_{kj}}{(s-d_k)^j} \leftrightarrow$

$$\begin{cases} A_{kj}\frac{t^{j-1}}{(j-1)!}(d_k)^n u[n], & ROC: Re(s) > d_k \\ -A_{kj}\frac{t^{j-1}}{(j-1)!}(d_k)^n u[-n-1], & ROC: Re(s) < d_k \end{cases}$$

where, $A_{kj} = \frac{1}{(r-j)!}\left[\frac{d^{r-j}}{ds^{r-j}}\frac{B'(s)}{A(s)}(s-d_k)^r\right]_{s=d_k}$.

■ D.2.1 Initial-Value Theorem

$$\lim_{s\to\infty} sX(s) = x(0^+)$$

This result does not apply to rational functions $X(s)$ in which the order of the numerator polynomial is equal to or greater than the order of the denominator polynomial. In that case, $X(s)$ would contain terms of the form cs^k, $k \ge 0$. Such terms correspond to the impulses and their derivatives located at time $t = 0$.

■ D.2.2 Final-Value Theorem

$$\lim_{s\to 0} sX(s) = \lim_{t\to\infty} x(t)$$

This result requires that all the poles of $sX(s)$ be in the left half of the s-plane.

■ D.2.3 Unilateral Differentiation Property, General Form

$$\frac{d^n}{dt^n}x(t) \xleftrightarrow{\mathcal{L}_u} s^nX(s) - \left.\frac{d^{n-1}}{dt^{n-1}}x(t)\right|_{t=0^-} - s\left.\frac{d^{n-2}}{dt^{n-2}}x(t)\right|_{t=0^-} - \cdots - s^{n-2}\left.\frac{d}{dt}x(t)\right|_{t=0^-} - s^{n-1}x(0^-)$$

Simulation Diagram:

$$\left[\sum_{j=0}^{n} a_j \frac{d^j y}{dt^j} = \sum_{j=0}^{n} b_j \frac{d^j x}{dt^j}\right] \Leftrightarrow H(s) = \frac{b_n s^n + \ldots + b_0}{a_n s^n + \ldots + a_0}.$$