Section 5.1
The Basics of Counting

THE RULE OF SUM

If A and B are disjoint sets then $|A \quad B| = |A| + |B|$

Example:

Suppose statement labels in a programming language must be a single letter or a single decimal digit.

Since a label cannot be both at the same time,

the number of labels

= the number of letters + the number of decimal digits

= 26 + 10 = 36.

THE RULE OF PRODUCT

$| A \times B | = | A | | B |$

Example:

- Statement labels in Basic can be either
  - a single letter or
  - a letter followed by a digit.

Find the number of possible labels.

We can partition the set of all labels *L* into the disjoint subsets consisting of

- the set of single letter labels *S*

and

- the set of single letters followed by a digit *D*

and

- $L = S \quad D$.

Use the rule of sum to compute the cardinality of *L* if we can compute the cardinality of *D*.

- The elements of D are ordered pairs of the form [*a*, *d*] where a is an alphabetic character and d is a digit.

- By the rule of product the cardinality of D is the product of the cardinality of the two sets:

- (the alphabetic characters)(the decimal digits)

  = (26)(10)

= 260.

The cardinality of L is 26 + 260 = 286.

THE PRINCIPLE OF INCLUSION-EXCLUSION

If A and B are not disjoint:

$$|A \quad B| = |A| + |B| - |A \cap B|$$

Don't count objects in the intersection of two sets more than once!

Example:

Count the number of bit strings of length 4 which begin with a 1 or end with a 00.

The set can be expressed as the union of

- the subset S of strings which begin with 1

and

- the subset O that end in 00.

Unfortunately the two subsets overlap.

• The cardinality of S is 8 (why?)

• The cardinality of O is 4 (why?).

Hence, by the exclusion-inclusion principle, the cardinality of the union is 12 minus the cardinality of the intersection.

How many strings are in the intersection?

Those strings that begin with 1 and end in 00 or 2 such strings.

The total number is 10 = 8 + 4 - 2.

Check:

• Strings in S that begin with 1:

1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111

• Strings O that end with 00:

0000, 0100, 1000, 1100

• 1000 and 1100 appear in both sets.

Count them once.

More Counting Examples:

Find the number of three-letter initials where none of the letters is repeated.

Apply the rule of product remembering that a letter cannot appear twice to get

(26)(25)(24).

Count the number of bit strings of length 4.

Apply the rule of product to get 2^4.

Count the number of bit strings of length 4 or less.

Apply the rule of sum to get the disjoint subsets of length 1, 2, 3 and 4.

Then apply the rule of product to count each subset to get

$$2 + 4 + 8 + 16 = 2^1 + 2^2 + 2^3 + 2^4.$$

Count the set S of 3 digit numbers which begin or end with an even digit.

Assume that 0 is even but a number cannot begin with 0.

The set is the union of the two subsets:

- The set B of three digit numbers that begin with 2, 4, 6 or 8.

This set has cardinality

$$(4)(10)(10).$$

(why?)

- The set C of three digit numbers that end with 0, 2, 4, 6, or 8 and do not begin with 0.

This set has cardinality

$$(5)(9)(10).$$

(why?)

- Now we use the inclusion-exclusion principle to eliminate the overlap of sets B and C.

Their intersection:

The 3 digit numbers that begin with 2, 4, 6, or 8 and end with 0, 2, 4, 6, or 8.

The intersection has the cardinality

$$(4)(10)(5)$$

Hence the cardinality is

$$400 + 450 - 200 = 650.$$
