Section 2.1
Sets

A ***set*** is a collection or group of objects or *elements* or *members*. (Cantor 1895)

• A set is said to *contain* its elements.

• There must be an underlying universal set U, either specifically stated or understood.

Notation:

• list the elements between braces:

$$S = \{a, b, c, d\}=\{b, c, a, d, d\}$$

(Note: listing an object more than once does not change the set. Ordering means nothing.)

• specification by predicates:

$$S= \{x| P(x)\},$$

S contains all the elements from U which make the predicate *P* true.

• brace notation with ellipses:

$$S = \{ . . . , -3, -2, -1\},$$

the negative integers.

Common Universal Sets

- R = reals
- N = natural numbers = {0,1, 2, 3, . . . }, the *counting* numbers
- Z = all integers = {. . , -3, -2, -1, 0, 1, 2, 3, 4, . .}
- Z^+ is the set of positive integers

Notation:

x is a member of S or x is an element of S:

x S.

x is not an element of S:

x S.

Subsets

Definition: The set A is a *subset* of the set B, denoted A B, iff

$$x[x \quad A \rightarrow x \quad B]$$

Definition: The *void* set, the *null* set, the *empty* set, denoted , is the set with no members.

Note: the assertion x is always false. Hence

$$x[x \quad \rightarrow x \quad B]$$

is always true(vacuously). Therefore, is a subset of every set.

Note: A set B is always a subset of itself.

Definition: If A B but A ≠ B the we say A is a *proper* subset of B, denoted $A \quad B$ (in some texts).

Definition: The set of all subset of a set A, denoted P(A), is called the *power set* of A.

Example: If A = {a, b} then

$$P(A) = \{ \quad , \{a\}, \{b\}, \{a,b\}\}$$

Definition: The number of (distinct) elements in A, denoted |A|, is called the *cardinality* of A.

If the cardinality is a natural number (in N), then the set is called *finite*, else *infinite*.

Example:

A = {a, b},

|{a, b}| = 2,

|P({a, b})| = 4.

A is finite and so is P(A).

Useful Fact: |A|=n implies |P(A)| = 2^n

N is infinite since |N| is not a natural number. It is called a *transfinite cardinal number*.

Note: Sets can be both members and subsets of other sets.

Example:

A = { ,{ }}.

A has two elements and hence four subsets:

, { }, {{ }}. { ,{ }}

Note that is both a member of A and a subset of A!

Russell's paradox: Let S be the set of all sets which are not members of themselves. Is S a member of itself?

Another paradox: Henry is a barber who shaves all people who do not shave themselves. Does Henry shave himself?

Definition: The *Cartesian product* of A with B, denoted A × B, is the set of ordered pairs {<a, b> | $a \quad A \quad b \quad B$}

Notation: $$\mathop{\times}_{i=1}^{n} A_i = \{< a_1, a_2, ..., a_n > | a_i \quad A_i\}$$

Note: The Cartesian product of anything with is . (why?)

Example:

A = {a,b}, B = {1, 2, 3}

AxB = {<a, 1>, <a, 2>, <a, 3>, <b, 1>, <b, 2>, <b, 3>}

What is BxA? AxBxA?

If |A| = m and |B| = n, what is |AxB|?
