## Finite Element Analysis

Finite element analysis is a numerical procedure that solves differential equations using integration procedures.

Consider the following differential equation:

$$D\frac{d^2 y}{dx^2} + Q = 0$$

Assume that we do not know the solution of this equation, but that we can make some rational decisions about an approximation to the solution.

Let the approximate value of y(x) be h(x).

Since an approximation has been made for y(x), there will be some error in the equation:

$$D\frac{d^2 h(x)}{dx^2} + Q = R(x)$$

R(x) is called the **residual function.**

The idea behind FEA is to find the solution to the original differential equation while minimizing the size of errors caused by approximations.

One of the more common integral methods used in FEA analysis is the **Galerkin Method**.

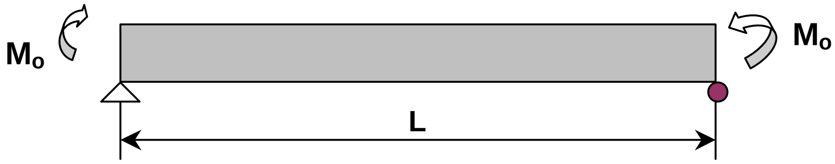
The integral equation that follows is typically the one solved to find solutions in FEA:

$$\int W_i(x)R(x)dx = 0$$

Where $W_i$ is called a **weighting function** and R(x) is the **residual function**.  To make the integral method the Galerkin method, the weighting function is whatever function was originally used as the approximate solution.  To find the residual function, the approximate solution is plugged into the differential equation given above.

An example will help clarify these concepts.

Consider the following beam:




The differential equation that describes this system is:

$$EI\frac{d^2y}{dx^2} - M_o = 0$$

An approximate solution to this problem is:

$$y(x) = A\sin(\frac{px}{L})$$

The approximate solution for deflection, y(x), holds at the boundaries (y = 0 at x = 0 and x = L).

The Galerkin method requires that we find the **residual function** first:

$$EI\frac{d^2\left(A\sin\left(\frac{px}{L}\right)\right)}{dx^2} - M_o = 0$$

$$EI\frac{p}{L}\frac{d}{dx}\left(A\cos\left(\frac{px}{L}\right)\right) - M_o = 0$$

$$-EI\frac{p^2}{L^2}A\sin\left(\frac{px}{L}\right) - M_o = 0$$

Next, **the weighting function**, $\mathbf{W_i(x)}$, is multiplied by the **residual function** and the result is integrated. In the Galerkin method, the weighting function is the approximate solution:

$$\int_0^L \sin\left(\frac{px}{L}\right)\left[-EI\frac{Ap^2}{L^2}\sin\left(\frac{px}{L}\right) - M_o\right]dx$$

$$\frac{-EIAp^{2}}{L^{2}}\left(-\sin\left(\frac{2px}{L}\right)+\frac{x}{2}\right)-\frac{L}{p}\cos\left(\frac{px}{L}\right)M_{o}$$

=

$$A=-\frac{4M_{o}L^{2}}{p^{3}EI}$$

$$y(x)=-\frac{4M_{o}L^{2}}{p^{3}EI}\sin\left(\frac{kx}{L}\right)$$

The exact solution is:

$$y(x)=\frac{M_{o}x}{EI}(x-L)$$

The steps involved in FEA are:

1. **Discretize** the region (locating and numbering nodes, and specifying nodal coordinate values)
2. Specify the approximation equation. An equation is written for each element.
3. Develop the system of equations (**stiffness matrix, boundary conditions**).
4. Solve system of equations.
5. Calculate quantities of interest (strains, stresses, temperatures, fluid velocities)