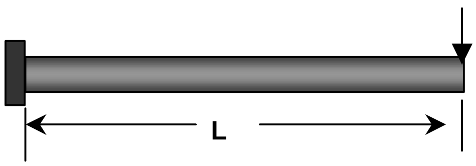
# Finite Element Analysis

Finite Element Analysis (FEA) is a **numerical procedure** that mechanical engineers use for many kinds of analysis:  stress and strain analysis, heat transfer analysis, vibrations analysis, etc.  In our class, we will use a package called COSMOS and we will perform stress and strain analysis.  First, however, a brief introduction to FEA will help you understand what COSMOS actually does.

Every part has some amount of "springiness"--even parts made of materials like steel, aluminum, and other metals.  This "springiness" depends on what kinds of material the part is made of, how big the part is, and how resistant to loading the part is.

For example, consider a cantilevered, steel beam of length L, with a rectangular cross-sectional area, A.

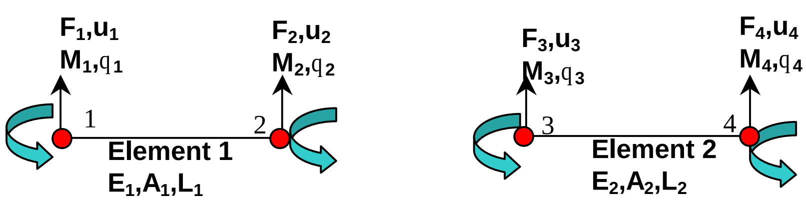



You will recall from your strength of materials class that the beam's deflection for this case is given as:

$$d = \frac{Fx^2}{6EI}(x - 3L)$$

We could use this formula to predict the deflection of the beam at any point along the beam, however we are going to solve this problem in a slightly different way and compare our results to the solution for deflection given above.

Let's begin by dividing the beam into two pieces and assuming that the free body diagram for each piece looks like the picture below.




- Each piece of the beam is called an **element** and each element possesses two **nodes**.  Element 1 has nodes 1 and 2, Element 2 has nodes 3 and 4.

- Each element has a cross section, Ai, length, Li, and modulus of elasticity, Ei (i indicates the element number)

- Each node "sees" a force, Fj, a moment, Mj, a slope, θj, and a displacement, uj, (j indicates the node number)

- Nodes 2 and 3 have the same force, same moment, same slope, and same displacement.

| | | |
|---|---|---|
| $F_2$ | = | $F_3$ |
| $M_2$ | = | $M_3$ |
| $\theta_2$ | = | $\theta_3$ |
| $u_2$ | = | $u_3$ |

We will use the method of **superposition** to determine the values of displacement, and slope of each node. We are going to treat each element as if it were a spring and use the formula, **K**d **= f**. In Matrix form:

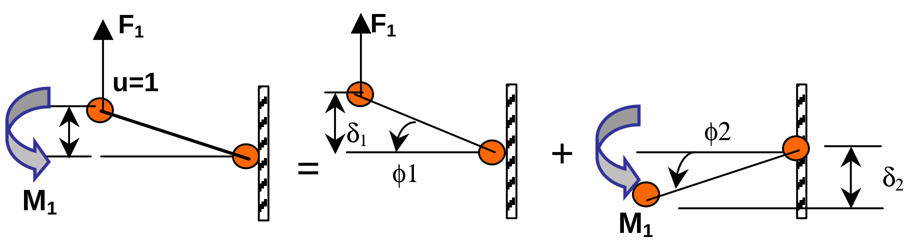
$$\begin{Bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ q_1 \\ u_2 \\ q_2 \end{Bmatrix}$$

First we will give the left most node, node 1, 1 unit of displacement, and hold all other displacements at 0.0

$$\begin{Bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$f_1 = k_{11}$$
$$M_1 = k_{21}$$
$$f_2 = k_{31}$$
$$M_2 = k_{41}$$

Here's what we have done, from a physical perspective:




What this picture says is that we can look at the influence of the force, $F_1$ on the slope and displacement, and the influence of the moment on the slope and displacement, add the results together, to get the total effect--superposition.

$$u_1 = 1 = d_1 + d_2 = \frac{F_1 L_1^{\ 3}}{3E_1 I_1} - \frac{M_1 L_1^{\ 2}}{2E_1 I_1}$$

$$q_1 = 0 = f_1 + f_2 = -\frac{F_1 L_1^{\ 2}}{2E_1 I_1} + \frac{M_1 L_1}{E_1 I_1}$$

Now solve these two equations for $F_1$, and $M_1$

$$\frac{1}{EI}\begin{bmatrix} \frac{L_1^{\ 3}}{3} & -\frac{L_1^{\ 2}}{2} \\ -\frac{L_1^{\ 2}}{2} & L_1 \end{bmatrix}\begin{Bmatrix} F_1 \\ M_1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\frac{1}{EI}\left\|\begin{bmatrix} \frac{L_1^{\ 3}}{3} & -\frac{L_1^{\ 2}}{2} \\ -\frac{L_1^{\ 2}}{2} & L_1 \end{bmatrix}\right\| = \frac{L_1^{\ 4}}{12E^2I^2}$$

$$\left\|\begin{bmatrix} 1 & -\frac{L_1^{\ 2}}{2EI} \\ 0 & \frac{L_1}{EI} \end{bmatrix}\right\| = \frac{L_1}{EI}$$

$$\left\|\begin{bmatrix} \frac{L_1^{\ 3}}{3EI} & 1 \\ -\frac{L_1^{\ 2}}{2EI} & 0 \end{bmatrix}\right\| = \frac{L_1^{\ 2}}{2EI}$$

$$F_1 = \frac{L_1}{\frac{L_1^{\ 4}EI}{12E^2I^2}} = \frac{12EI}{L_1^{\ 3}} = k_{11}$$

$$M_1 = \frac{\frac{L_1^{\ 2}}{2EI}}{\frac{L_1^{\ 4}}{12E^2I^2}} = \frac{6EI}{L_1^{\ 2}} = k_{21}$$

Now we can finish Element 1 by using the equations of equilibrium:

$$F_1 + F_2 = 0; F_2 = -F_1$$

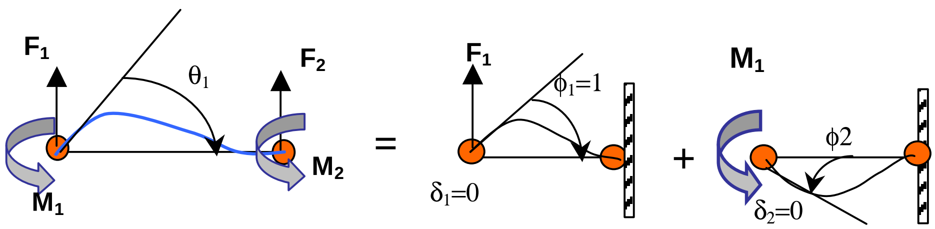
$$F_2 = -\frac{12EI}{L_1^{\ 3}} = k_{31}$$

$$\sum M_{node\ 2} = M_1 + M_2 - L_1F_1 = 0 = \frac{6EI}{L_1^{\ 2}} + M_2 - L_1\frac{12EI}{L_1^{\ 3}}$$

$$M_2 = -\frac{6EI}{L_1^{\ 2}} + \frac{12EI}{L_1^{\ 2}} = \frac{6EI}{L_1^{\ 2}} = k_{41}$$

$$\begin{Bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & k_{12} & k_{13} & k_{14} \\ \frac{6EI}{L^2} & k_{22} & k_{23} & k_{24} \\ -\frac{12EI}{L^3} & k_{32} & k_{33} & k_{34} \\ \frac{6EI}{L^2} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ q_1 \\ u_2 \\ q_2 \end{Bmatrix}$$

The next column of values in the **stiffness matrix** can be determined by setting the slope at node 1 equal to one unit, and forcing everything else to remain fixed.




$$u_1 = 0 = d_1 + d_2 = \frac{F_1 L_1^{\,3}}{3E_1 I_1} - \frac{M_1 L_1^{\,2}}{2E_1 I_1}$$

$$q_1 = 1 = f_1 + f_2 = -\frac{F_1 L_1^{\,2}}{2E_1 I_1} + \frac{M_1 L_1}{E_1 I_1}$$

$$\begin{vmatrix} 0 & -\frac{L_1^{\,2}}{2EI} \\ 1 & \frac{L_1}{EI} \end{vmatrix} = \frac{L_1^{\,2}}{2EI} \qquad \frac{1}{EI} \begin{bmatrix} \frac{L_1^{\,3}}{3} & -\frac{L_1^{\,2}}{2} \\ -\frac{L_1^{\,2}}{2} & L_1 \end{bmatrix} \begin{Bmatrix} F_1 \\ M_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\begin{vmatrix} \frac{L_1^{\,3}}{3EI} & 0 \\ -\frac{L_1^{\,2}}{2EI} & 1 \end{vmatrix} = \frac{L_1^{\,3}}{3EI} \qquad \frac{1}{EI} \begin{vmatrix} \frac{L_1^{\,3}}{3} & -\frac{L_1^{\,2}}{2} \\ -\frac{L_1^{\,2}}{2} & L_1 \end{vmatrix} = \frac{L_1^{\,4}}{12E^2 I^2}$$

$$F_1 = \frac{\frac{L_1^{\,2}}{2EI}}{\frac{L_1^{\,4}}{12(EI)^2}} = \frac{6EI}{L_1^{\,2}}$$

$$M_1 = \frac{\frac{L_1^{\,3}}{3EI}}{\frac{L_1^{\,4}}{12(EI)^2}} = \frac{4EI}{L_1}$$

Now, solve for $F_2$ and $M_2$ using equations of equilibrium

$$F_2 + F_1 = 0$$

$$-F_2 = F_1 = -\frac{6EI}{L_1^2}$$

$$M_1 + M_2 - (F_1 L_1) = 0$$

$$= \frac{6EI}{L_1} - \frac{4EI}{L_1} = \frac{2EI}{L_1}$$

Now the stiffness matrix looks like the following:

$$\begin{Bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L_1^3} & \frac{6EI}{L_1^2} & k_{13} & k_{14} \\ \frac{6EI}{L_1^2} & \frac{4EI}{L_1} & k_{23} & k_{24} \\ -\frac{12EI}{L_1^3} & -\frac{6EI}{L_1^2} & k_{33} & k_{34} \\ \frac{6EI}{L_1^2} & \frac{2EI}{L_1} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ q_1 \\ u_2 \\ q_2 \end{Bmatrix}$$

You should finish this problem using superposition and show that column 3 can be determined by setting $u_1$, $\theta_1$, and $\theta_2$ to zero and setting $u_2$ to 1. Column 4 can be determined by setting $\theta_2$ to 1 and all other nodal displacements to 0.0. The final matrix will look like this:

$$\begin{Bmatrix} f_1 \\ M_1 \\ f_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L_1^3} & \frac{6EI}{L_1^2} & -\frac{12EI}{L_1^3} & \frac{6EI}{L_1^2} \\ \frac{6EI}{L_1^2} & \frac{4EI}{L_1} & -\frac{6EI}{L_1^2} & \frac{2EI}{L_1} \\ -\frac{12EI}{L_1^3} & -\frac{6EI}{L_1^2} & \frac{12EI}{L_1^3} & -\frac{6EI}{L_1^2} \\ \frac{6EI}{L_1^2} & \frac{2EI}{L_1} & -\frac{6EI}{L_1^2} & \frac{4EI}{L_1} \end{bmatrix} \begin{Bmatrix} u_1 \\ q_1 \\ u_2 \\ q_2 \end{Bmatrix}$$

$$= \frac{EI}{L_1^3} \begin{bmatrix} 12 & 6L_1 & -12 & 6L_1 \\ 6L_1 & 4L_1^2 & -6L_1 & 2L_1^2 \\ -12 & -6L_1 & 12 & -6L_1 \\ 6L_1 & 2L_1^2 & -6L_1 & 4L_1^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ q_1 \\ u_2 \\ q_2 \end{Bmatrix}$$

This takes care of element 1. How about element 2? It turns out the stiffness matrix will look exactly the same, but $L_1$ becomes $L_2$, $E_1$ becomes $E_2$ and nodal forces and displacements also assume their new indicies (2 and 3).

Finite Element software completes this step for each element in the model. Then the elemental stiffness matrices are formed into a single matrix called the Global Stiffness Matrix in a process called **assembly**.

Once the global stiffness matrix is determined, and nodal loadings applied, the stiffness matrix is "inverted" and slopes and deflections are determined. Following this stage of analysis, a process called "post processing" is executed; post-processing uses deflection information to determine stresses.

$$e_x = \frac{s_x}{E} - u\frac{s_y}{E} + e_{xo}$$

$$e_y = -u\frac{s_x}{E} + \frac{s_y}{E} + e_{yo}$$

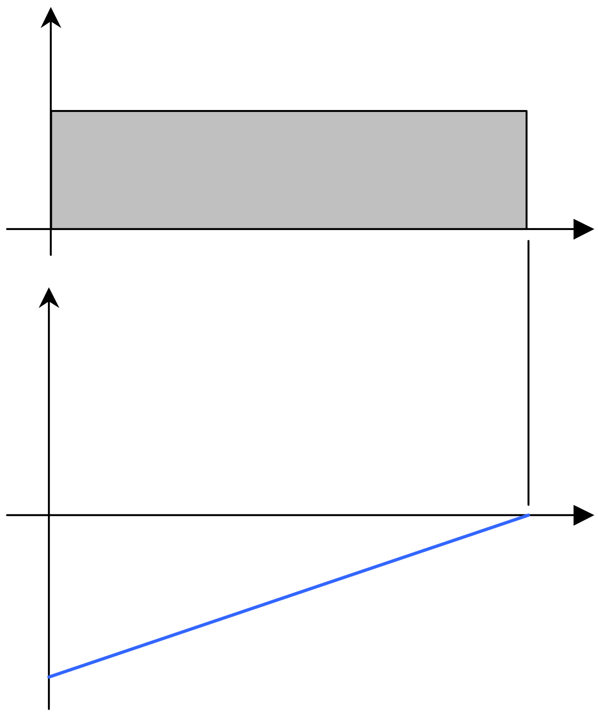
$$g_{xy} = \frac{2(1+u)t_{xy}}{E} + e_{xy}$$

You will recall that this particular problem is described by the following differential equation:

$$EI\frac{d^2u}{dx^2}+b=0$$

E is the beam's modulus of elasticity and I is the second moment of area. The objective in solving this differential equation is to find a function, u(x), that will enable us to find the deflection at any point along the beam.

To begin solution of this problem, let's look at the shear and moment diagrams for the cantilevered beam.



At x = 0, or at the left side of the beam, we notice that the shear force is + F, the moment is -FL (Mo), and although we did not draw the slope diagram, we know that the slope at the wall is 0.0 . It is also clear that the moment varies along the distance of the beam, unitl it becomes 0.0 at L.

$$EI\frac{d^2u}{dx^2}=Vx+M_o$$

V* x is the moment function as it varies from 0 → L.

Mathematically, we would express the boundary conditions on this problem like this:

$$EI\frac{d^2u}{dx^2}|_{x=0} = M_o$$
$$EI\frac{du}{dx}|_{x=0} = 0$$
$$u(x)|_{x=0} = 0$$

Moment at the wall is -FL, $M_o$
Slope at the wall is 0.0
Deflection at the wall is 0.0.

We can use this information to solve the differential equation.

$$EI\frac{d^2u}{dx^2} = Vx + M_o$$

$$\frac{d^2u}{dx^2} = \frac{1}{EI}(Vx + M_o)$$
$$\frac{du}{dx} = \int \frac{1}{EI}(Vx + M_o)dx = \frac{1}{EI}\left(V\frac{x^2}{2} + M_o x\right) + C_1$$
$$u(x) = \int\left[\frac{1}{EI}\left(V\frac{x^2}{2} + M_o x\right) + C_1\right]dx = \frac{1}{EI}\left(V\frac{x^3}{6} + M_o\frac{x^2}{2}\right) + C_1 x + C_2$$

Knowing that, $u(x)|_{x=0} = 0$, then we can show that $C_2$ is 0.0

And, knowing that $EI\frac{du}{dx}|_{x=0} = 0$, we can show that $C_1$ is 0.0.

So, the solution for u(x) is

$$\frac{1}{EI}\left(V\frac{x^3}{6} + M_o\frac{x^2}{2}\right)$$
$$V = F$$
$$M_o = -FL$$
$$u(x) = \frac{Fx^2}{6EI}(x - 3L)$$

So we have shown where the solution for deflection of a cantilevered beam comes from--what does this have to do with FEA?

The FEA software, regardless of what software it is, does not know what kind of problem you are solving. The only thing the software can do is solve differential equations (or minimize functionals). So, how does it work?